On soliton-preserving anti-self-dual Yang-Mills reductions

Shangshuai Li

This is a joint work with K.I. Maruno and D.J. Zhang

March 22nd, 2026

Yamagata Integrable System Workshop

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The ASDYM equations

The ASDYM equations describe interactions in the gauge field with anti-self-duality, they have the following properties:

(1) They are 4-d integrable systems

(2) They possess soliton structure of Wronskian or Grammian

(3) They can be reduced to many (2+1)-d equations

Ward's conjecture [Ward-1985]

... many (and perhaps all?) of the ordinary and partial differential equations that are regarded as being integrable or solvable may be obtained from the self-dual gauge field equations (or its generalizations) by reduction.

Our goal: reduce 4-d ASDYM to some (2+1)-d integrable equations, while the soliton structure can be preserved.

The ASDYM equations

Let G be the gauge group and $\mathfrak{g}$ be its associate Lie algebra. Let $(z, \tilde{z}, w, \tilde{w})$ be complexified space-time coordinates and $A_z, A_w \in \mathfrak{g}$ be gauge potentials, the ASDYM equations read as:

$$[\partial_z - \lambda\partial_{\tilde{w}} + A_z, \partial_w - \lambda\partial_{\tilde{z}} + A_w] = 0,$$

where the gauge potentials read as:

$$A_z = \partial_{\tilde{w}}K = -(\partial_z J)J^{-1}, \qquad A_w = \partial_{\tilde{z}}K = -(\partial_w J)J^{-1}.$$

The compatible condition leads to two equations:

(i) $\partial_{\tilde{z}}((\partial_z J)J^{-1}) - \partial_{\tilde{w}}((\partial_w J)J^{-1}) = 0$ (Yang eqn)

(ii) $\partial_z\partial_{\tilde{z}}K - \partial_w\partial_{\tilde{w}}K - [\partial_{\tilde{z}}K, \partial_{\tilde{w}}K] = 0$ (Chalmers-Siegel eqn)

Both of them are regarded as ASDYM equations.

Darboux transformation

Vectorial BDT of ASDYM [**S.Li**-Hamanaka-Huang-Zhang-2025]

Assume $J^{[0]}$ and $K^{[0]}$ to be seed solutions that satisfy

$$A_z^{[0]} = \partial_{\tilde{w}} K^{[0]} = -(\partial_z J^{[0]})(J^{[0]})^{-1},$$
$$A_w^{[0]} = \partial_{\tilde{z}} K^{[0]} = -(\partial_w J^{[0]})(J^{[0]})^{-1}.$$

Suppose $\eta \in \mathbb{C}_{N\times 2}$ and $\theta \in \mathbb{C}_{2\times N}$ satisfy the linear system

$$\partial_z \eta - \Xi(\partial_{\tilde{w}}\eta) = \eta A_z^{[0]}, \qquad \partial_z \theta - (\partial_{\tilde{w}}\theta)\Lambda = -A_z^{[0]}\theta,$$
$$\partial_w \eta - \Xi(\partial_{\tilde{z}}\eta) = \eta A_w^{[0]}, \qquad \partial_w \theta - (\partial_{\tilde{z}}\theta)\Lambda = -A_w^{[0]}\theta,$$

where $\Xi, \Lambda \in \mathbb{C}_{N\times N}$ are spectral parameter matrices.

Darboux transformation

Vectorial BDT of ASDYM [LHHZ-2025]

Then we introduce Ω determined by associated Sylvester equation

$$\Xi\Omega - \Omega\Lambda = \eta\theta.$$

Finally, the vectorial BDT of ASDYM is given by

$$J = (I - \theta\Omega^{-1}\Xi^{-1}\eta)J^{[0]}, \qquad K = \theta\Omega^{-1}\eta + K^{[0]}.$$

The vectorial BDT of ASDYM transforms seed solutions to N-soliton solutions with different backgrounds.

Remark: By setting $A_z^{[0]} = A_w^{[0]} = 0$, one can extract Cauchy matrix structure from it, which leads to ASDYM bright N-soliton.

Soliton solutions of ASDYM

Let $J^{[0]} = I$ and $K^{[0]} = 0$, then $A_z^{[0]} = A_w^{[0]} = 0$, and one obtains:

Cauchy matrix structure [**S.Li**-Qu-Zhang-2023,LHHZ-2025]

Sylvester equation:

$$\Xi\Omega - \Omega\Lambda = \eta\theta.$$

Dispersion relations:

$$\partial_z\eta = \Xi(\partial_{\tilde{w}}\eta), \qquad \partial_z\theta = (\partial_{\tilde{w}}\theta)\Lambda,$$
$$\partial_w\eta = \Xi(\partial_{\tilde{z}}\eta), \qquad \partial_w\theta = (\partial_{\tilde{z}}\theta)\Lambda,$$

Explicit solutions:

$$J = I - \theta\Omega^{-1}\Xi^{-1}\eta, \qquad K = \theta\Omega^{-1}\eta.$$

Soliton solutions of ASDYM

Suppose the spectral parameter matrices are both diagonal, then the Cauchy matrix structure is satisfied by the following construction [Ohta-2024]:

$$\Xi = \mathrm{Diag}(\xi_1, \cdots, \xi_N), \qquad \eta = (\phi_{is}(\xi_i z + \tilde{w}, \xi_i w + \tilde{z}))_{N\times 2},$$
$$\Lambda = \mathrm{Diag}(\lambda_1, \cdots, \lambda_N), \qquad \theta = (\psi_{sj}(\lambda_j z + \tilde{w}, \lambda_j w + \tilde{z}))_{2\times N},$$
$$\Omega = (\Omega_{ij})_{N\times N}, \qquad \Omega_{ij} = \frac{\sum_{s=1}^{2} \phi_{is}\psi_{sj}}{\xi_i - \lambda_j},$$

where ϕ_{is} and ψ_{sj} are arbitrary functions. If ϕ_{is} and ψ_{sj} are exponential functions, it gives rise to soliton solution:

$$\phi_{is} = A_{is}\exp(\alpha_{is}(\xi_i z + \tilde{w}) + \beta_{is}(\xi_i w + \tilde{z})), \tag{1a}$$
$$\psi_{sj} = B_{sj}\exp(\gamma_{sj}(\lambda_j z + \tilde{w}) + \delta_{sj}(\lambda_j w + \tilde{z})). \tag{1b}$$

Reduction to (2+1)-d integrable systems

Let $G = GL(2)$, the J-matrix and K-matrix can be decomposed to

$$J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}, \qquad K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}.$$

To reduce 4-d system to (2+1)-d system, and without losing any generalities, we assume the $\tilde{w}$-derivative can be eliminated after applying the following condition:

$$\partial_{\tilde{w}} K = [K, \frac{1}{2}\sigma_3], \qquad \sigma_3 := Diag(1, -1). \tag{2}$$

Remark: It can be achieved by following setting in (1)

$$\alpha_{i1} = \frac{1}{2}, \qquad \alpha_{i2} = -\frac{1}{2}, \qquad \gamma_{1j} = -\frac{1}{2}, \qquad \gamma_{2j} = \frac{1}{2},$$

The (2+1)-d NLS equation

In this sense, the Chalmers-Siegel equation simplifies to

$$\partial_z\partial_{\tilde{z}}K - [\partial_w K, \frac{1}{2}\sigma_3] - [\partial_{\tilde{z}}K, [K, \frac{1}{2}\sigma_3]] = 0.$$

Expanding it in terms of matrix entries, one obtains

$$\begin{aligned} K_{11} &= \partial_z^{-1}(K_{12}K_{21}), \qquad K_{22} = -\partial_z^{-1}(K_{21}K_{12}), \\ \partial_z\partial_{\tilde{z}}K_{12} &= -\partial_w K_{12} - (\partial_{\tilde{z}}K_{11})K_{12} + K_{12}(\partial_{\tilde{z}}K_{22}), \\ \partial_z\partial_{\tilde{z}}K_{21} &= \partial_w K_{21} + (\partial_{\tilde{z}}K_{22})K_{21} - K_{21}(\partial_{\tilde{z}}K_{11}). \end{aligned}$$

Defining $(r, q) := (K_{21}, -K_{12})$, it yields the (2+1)-d NLS system:

$$r_w = r_{z\tilde{z}} - 2r\partial_z^{-1}(qr)_{\tilde{z}}, \qquad \text{(NLS-1)}$$

$$q_w = -q_{z\tilde{z}} + 2q\partial_z^{-1}(rq)_{\tilde{z}}, \qquad \text{(NLS-2)}$$

The (2+1)-d NLS equation

Define $(x, y, t) := (\mathrm{i}z, \mathrm{i}\tilde{z}, \mathrm{i}\tilde{w})$, and introduce $u := r = q^*$, then (2+1)-d NLS system becomes the (2+1)-d NLS equation:

$$\mathrm{i}u_t + u_{xy} + 2u\partial_x^{-1}(|u|^2)_y = 0.$$

Remark: Similar reductions are revealed in KP reduction, relative ASDYM reduction are discussed [Kakei-Ikeda-Takasaki-2001].

Remark: The bilinear NLS can be obtained from bilinear ASDYM [Sasa-Ohta-Matsukidaira-1998].

From 4-d ASDYM to (2+1)-d NLS

$$\text{4-d ASDYM equation} \xrightarrow[(r,q) := (K_{21}, -K_{12})]{\partial_{\tilde{w}}K = [K, \frac{1}{2}\sigma_3]} \text{(2+1)-d NLS system}$$

The (2+1)-d NLS equation

Since

$$K = \theta\Omega^{-1}\eta = \begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix}\Omega^{-1}\begin{pmatrix}\eta_1 & \eta_2\end{pmatrix} = \begin{pmatrix}K_{11} & K_{12} \\ K_{21} & K_{22}\end{pmatrix},$$

thus we have

$$q = -K_{12} = -\theta_1\Omega^{-1}\eta_2, \qquad r = K_{21} = \theta_2\Omega^{-1}\eta_1.$$

The conjugate reduction is given by $(A^\dagger = (A^*)^T)$:

$$\Lambda = \Xi^\dagger, \qquad \theta_1 = \eta_1^\dagger, \qquad \theta_2 = \eta_2^\dagger, \qquad \Omega = -\Omega^\dagger,$$

which leads to

$$q = -\eta_1^\dagger\Omega^{-1}\eta_2 = (\eta_2^\dagger\Omega^{-1}\eta_1)^\dagger = r^*.$$

The (2+1)-d GI equation

Reduction to (2+1)-d GI (Gerdjikov-Ivanov) is based on the result of (2+1)-d NLS. Starting from (2), we have

$$(\partial_z J)J^{-1} = \partial_{\tilde{w}} K = [K, \frac{1}{2}\sigma_3],$$

which indicates

$$\begin{pmatrix} \partial_z J_{11} & \partial_z J_{12} \\ \partial_z J_{21} & \partial_z J_{22} \end{pmatrix} = \begin{pmatrix} K_{12} J_{21} & K_{12} J_{22} \\ -K_{21} J_{11} & -K_{21} J_{12} \end{pmatrix}.$$

Let $(p, q, r) = (J_{21} J_{11}^{-1}, -K_{12}, K_{21})$, they yield the relation:

$$p_z = -r + p^2 q. \tag{MT}$$

This Miura Transformation links NLS (r, q) and GI (p, q)!

The (2+1)-d GI equation

From the second equation of NLS system

$$q_w = -q_{z\tilde{z}} + 2q\partial_z^{-1}(rq)_{\tilde{z}}, \qquad \text{(NLS-2)}$$

we substitute $r = -p_z + p^2 q$ into it, which gives rise to

$$q_w = -q_{z\tilde{z}} - 2q\partial_z^{-1}(p_z q)_{\tilde{z}} + 2q\partial_z^{-1}(p^2 q^2)_{\tilde{z}}. \qquad \text{(GI-2)}$$

On the other hand, the w-derivative of (MT) shows

$$r_w = (2pq - \partial_z)p_w + p^2 q_w,$$

which along with (NLS-1) and (NLS-2) indicates an equation of:

$$p_w = p_{z\tilde{z}} - 2p\partial_z^{-1}(pq_z)_{\tilde{z}} - 2p\partial_z^{-1}(p^2 q^2)_{\tilde{z}}. \qquad \text{(GI-1)}$$

The (2+1)-d GI equation

Define $(x, y, t) := (\mathrm{i}z, \mathrm{i}\tilde{z}, \mathrm{i}\tilde{w})$, and introduce $u := p = q^*$, then (2+1)-d GI system becomes the (2+1)-d GI equation:

$$\mathrm{i}u_t + u_{xy} + 2\mathrm{i}u\partial_x^{-1}(uu_x^*)_y + 2u\partial_x^{-1}(|u|^4)_y = 0.$$

The relation between ASDYM and NLS and GI are summarized:

From 4-d ASDYM to (2+1)-d GI

4-d ASDYM equation $\xrightarrow[(r, q) := (K_{21}, -K_{12})]{\partial_{\tilde{w}}K = [K, \frac{1}{2}\sigma_3]}$ (2+1)-d NLS system

4-d ASDYM equation $\downarrow$ $(p, q) := (J_{21}J_{11}^{-1}, -K_{12})$ (2+1)-d GI system

(2+1)-d NLS system $\downarrow$ $p := J_{21}J_{11}^{-1}$ Miura transformation

(2+1)-d GI system $\xleftarrow{r = -p_z + p^2 q}$ Miura transformation

The (2+1)-d GDNLS equation

Other than GI equation, the DNLS equations are known to have three types, the Kaup-Newell type and the Chen-Lee-Liu type. They are linked through gauge transformation [Wadati-Sogo-1983]. Moreover, the three equations can be viewed in one generalized DNLS equation [Kundu-1984,1987].

Gauge transformation and GDNLS equation

Let $u_{[\mathrm{GDNLS}]} = u_{[\mathrm{GI}]} s^{\gamma}$, $s = \exp(\mathrm{i}\partial_x^{-1}|u_{[\mathrm{GI}]}|^2)$, then $u_{[\mathrm{GDNLS}]}$ yields:

$$\begin{aligned} &\mathrm{i}u_t + u_{xy} - \mathrm{i}\gamma u\partial_x^{-1}(u_x u^*)_y - \mathrm{i}\gamma u_x\partial_x^{-1}(|u|^2)_y \qquad \text{(GDNLS)} \\ &\quad + 2\mathrm{i}(\gamma-1)u\partial_x^{-1}(uu_x^*)_y + (\gamma-1)(\gamma-2)u\partial_x^{-1}(|u|^4)_y = 0. \end{aligned}$$

When $\gamma = 0, 1, 2$, (GDNLS) becomes GI, CLL, KN, respectively.

The (2+1)-d GDNLS equation

Realization of gauge factor in ASDYM reduction:

$$s = \exp(-\partial_z^{-1}(qp)) = \exp(\partial_z^{-1}(\ln(J_{11})_z)) = J_{11}.$$

Construction of (2+1)-d GDNLS: Suppose (p, q) satisfies (2+1)-d GI system, by defining $(u, v) = (ps^{\gamma}, qs^{-\gamma})$, it yields

$$\begin{aligned} u_w = {} & u_{z\tilde{z}} + \gamma u \partial_z^{-1}(u_z v)_{\tilde{z}} + \gamma u_z \partial_z^{-1}(uv)_{\tilde{z}} && \text{(GDNLS-1)} \\ & + 2(\gamma-1)u\partial_z^{-1}(uv_z)_{\tilde{z}} - (\gamma-1)(\gamma-2)u\partial_z^{-1}(u^2v^2)_{\tilde{z}}, \\ v_w = {} & -v_{z\tilde{z}} + \gamma v \partial_z^{-1}(uv_z)_{\tilde{z}} + \gamma v_z \partial_z^{-1}(uv)_{\tilde{z}} && \text{(GDNLS-2)} \\ & + 2(\gamma-1)v\partial_z^{-1}(u_z v)_{\tilde{z}} + (\gamma-1)(\gamma-2)v\partial_z^{-1}(u^2v^2)_{\tilde{z}}. \end{aligned}$$

Define $(x, y, t) := (\mathrm{i}z, \mathrm{i}\tilde{z}, \mathrm{i}\tilde{w})$, and assume $v = u^*$, the (2+1)-d GDNLS system becomes (GDNLS).

The (2+1)-d GDNLS equation

Let $\gamma = 1$, defining $(\tilde{p}, \tilde{q}) = (ps, qs^{-1})$, it yields (2+1)-d CLL:

$$\tilde{p}_w = \tilde{p}_{z\tilde{z}} + \tilde{p}\partial_z^{-1}(\tilde{p}_z\tilde{q})_{\tilde{z}} + \tilde{p}_z\partial_z^{-1}(\tilde{p}\tilde{q})_{\tilde{z}}, \quad \text{(CLL-1)}$$

$$\tilde{q}_w = -\tilde{q}_{z\tilde{z}} + \tilde{q}\partial_z^{-1}(\tilde{p}\tilde{q}_z)_{\tilde{z}} + \tilde{q}_z\partial_z^{-1}(\tilde{p}\tilde{q})_{\tilde{z}}. \quad \text{(CLL-2)}$$

Let $\gamma = 2$, defining $(\hat{p}, \hat{q}) = (ps^2, qs^{-2})$, it yields (2+1)-d KN:

$$\hat{p}_w = \hat{p}_{z\tilde{z}} + 2(\hat{p}\partial_z^{-1}(\hat{p}\hat{q})_{\tilde{z}})_z, \quad \text{(KN-1)}$$

$$\hat{q}_w = -\hat{q}_{z\tilde{z}} + 2(\hat{q}\partial_z^{-1}(\hat{p}\hat{q})_{\tilde{z}})_z. \quad \text{(KN-2)}$$

The construction of three DNLS equations are summarized:

Realization of DNLS in ASDYM reduction

- (2+1)-d **GI**: $(p, q) = (J_{21}J_{11}^{-1}, -K_{12})$
- (2+1)-d **CLL**: $(\tilde{p}, \tilde{q}) = (ps, qs^{-1}) = (J_{21}, -J_{11}^{-1}K_{12})$
- (2+1)-d **KN**: $(\hat{p}, \hat{q}) = (ps^2, qs^{-2}) = (J_{21}J_{11}, -J_{11}^{-2}K_{12})$

The (2+1)-d GDNLS equation

The relation between ASDYM and GI and GDNLS are summarized:

From 4-d ASDYM to (2+1)-d GDNLS

$$\begin{array}{ccc}
\text{4-d ASDYM equation} & \xrightarrow[(J_{21}J_{11}^{-1},\, -K_{12})]{(p,q) :=} & \text{(2+1)-d GI system} \\
\Big\downarrow (u,v) := (J_{21}J_{11}^{\gamma-1},\, -J_{11}^{-\gamma}K_{12}) & & \Big\downarrow s = e^{-\int (pq) dz} \\
\text{(2+1)-d GDNLS system} & \xleftarrow{(u,v) := (ps^{\gamma},\, qs^{-\gamma})} & \text{gauge factor } s = J_{11}
\end{array}$$

The (2+1)-d GDNLS equation

To achieve the conjugate condition of GDNLS (u, v), we first consider the case of GI (p, q), where:

$$p = \frac{J_{21}}{J_{11}} = -\frac{\theta_2\Omega^{-1}\Xi^{-1}\eta_1}{1-\theta_1\Omega^{-1}\Xi^{-1}\eta_1}, \qquad q = -K_{12} = -\theta_1\Omega^{-1}\eta_2.$$

The conjugate condition is given by [**S.Li**-Liu-Zhang-2025]:

$$\Lambda = \Xi^{\dagger}, \quad \theta_1 = \eta_1^{\dagger}, \quad \theta_2 = -\eta_2^{\dagger}\Xi^{\dagger}, \quad \Xi\Omega + \Omega^{\dagger}\Xi^{\dagger} = \eta_1\eta_1^{\dagger}.$$

Through a direct calculation we have $p = q^*$:

$$K_{12}^{\dagger}J_{11} = \eta_2^{\dagger}\Omega^{-\dagger}\eta_1 - \eta_2^{\dagger}(\Omega^{-\dagger}\Xi + \Xi^{\dagger}\Omega^{-1})\Xi^{-1}\eta_1 = -J_{21},$$

The (2+1)-d GDNLS equation

As for the gauge factor $s = J_{11}$, it yields $s^* = s^{-1}$:

$$\begin{aligned} J_{11} J_{11}^\dagger &= (1 - \eta_1^\dagger \Omega^{-1} \Xi^{-1} \eta_1)(1 - \eta_1^\dagger \Xi^{-\dagger} \Omega^{-\dagger} \eta_1) \\ &= 1 - \eta_1^\dagger (\Omega^{-1} \Xi^{-1} + \Xi^{-\dagger} \Omega^{-\dagger}) \eta_1 + \eta_1^\dagger \Omega^{-1} \Xi^{-1} \eta_1 \eta_1^\dagger \Xi^{-\dagger} \Omega^{-\dagger} \eta \\ &= 1 - \eta_1^\dagger (\Omega^{-1} \Xi^{-1} + \Xi^{-\dagger} \Omega^{-\dagger} - \Omega^{-1} \Xi^{-1} - \Xi^{-\dagger} \Omega^{-\dagger}) \eta_1 = 1. \end{aligned}$$

Thus $v^* = (s^{-\gamma} q)^* = s^\gamma p = u$, and it solves the GDNLS equation.

The (2+1)-d GDNLS equation

The relations between ASDYM, NLS and KN were revealed by using bilinear transformation [SOM-1998]:

- **(2+1)-d NLS:** $\mathrm{i}\partial_T\psi + \partial_X\partial_Y\psi + 2\psi\partial_X^{-1}\partial_Y|\psi|^2 = 0$

$$\psi = \frac{\tau_{n-1,m-1}}{\tau_{n,m}}, \qquad \psi^* = \frac{\tau_{n+1,m+1}}{\tau_{n,m}}. \qquad \text{(BT-NLS)}$$

- **(2+1)-d KN:** $\mathrm{i}\partial_T\psi + \partial_X\partial_Y\psi + 2\mathrm{i}\partial_X[\psi\partial_X^{-1}\partial_Y|\psi|^2] = 0$

$$\psi^* = \frac{-\mathrm{i}\tau_{n-1,m}\tau_{n,m+1}}{\tau_{n,m}^2}, \qquad \psi = \frac{\tau_{n,m}\tau_{n+1,m+1}}{\tau_{n,m+1}^2}. \qquad \text{(BT-KN)}$$

Remark: The above notations (equations) are form the original paper of [SOM-1998], we are interested in their bilinear forms.

The (2+1)-d GDNLS equation

The J-matrix and K-matrix have their bilinear transformations [Ohta-2024,LLZ-2025]:

$$J_{ij} = g_{ij}/f, \qquad K_{ij} = h_{ij}/f, \qquad 1 \leq i, j \leq 2.$$

In this sense, we have

- **BT of NLS from** $(r, q) = (K_{21}, -K_{12})$**:**

$$r = \frac{h_{21}}{f}, \qquad q = -\frac{h_{12}}{f}. \qquad \text{(BT-NLS-new)}$$

- **BT of KN from** $(\hat{p}, \hat{q}) = (J_{21}J_{11}, -J_{11}^{-2}K_{12})$**:**

$$\hat{p} = \frac{g_{11}g_{21}}{f^2}, \qquad \hat{q} = -\frac{fh_{12}}{g_{11}^2}. \qquad \text{(BT-KN-new)}$$

The (2+1)-d GDNLS equation

The BT of GDNLS are given in [Kakei-Sasa-Satsuma-1995]:

- **BT of GDNLS (KSS version):**

$$v = \frac{\tilde{f}^{\gamma-1}\tilde{g}}{f^{\gamma}}, \qquad u = \frac{f^{\gamma-1}g}{\tilde{f}^{\gamma}} \qquad \text{(BT-GDNLS)}$$

- **BT of GDNLS from** $(u, v) = (J_{21}J_{11}^{\gamma-1}, -J_{11}^{-\gamma}K_{12})$**:**

$$u = \frac{g_{11}^{\gamma-1}g_{21}}{f^{\gamma}}, \qquad v = -\frac{f^{\gamma-1}h_{12}}{g_{11}^{\gamma}}. \qquad \text{(BT-GDNLS-new)}$$

The BT of GDNLS from ASDYM yields KSS construction, and also gives rise to the BTs of GI, CLL, KN and FL (Fokas-Lenells) [Nakamura-Chen-1980,Liu-Wang-Zhang-2022].

Concluding remarks

Starting from J-matrix and K-matrix of GL(2) ASDYM equations, the structure of (2+1)-d NLS and GDNLS are clear.

Main theorem of this talk

Suppose $\Xi, \Lambda, \Omega, \eta = (\eta_1, \eta_2), \theta^T = (\theta_1^T, \theta_2^T)$ satisfy the Cauchy matrix structure of ASDYM [SQZ-2023], define

$$J_{11} = 1 - \theta_1 \Omega^{-1} \Xi^{-1} \eta_1, \qquad J_{21} = -\theta_2 \Omega^{-1} \Xi^{-1} \eta_1,$$
$$K_{12} = \theta_1 \Omega^{-1} \eta_2, \qquad K_{21} = \theta_2 \Omega^{-1} \eta_1.$$

Then NLS, GI and GDNLS are solved by ASDYM solitons:

- (2+1)-d **NLS**: $(K_{21}, -K_{12})$
- (2+1)-d **GI**: $(J_{21} J_{11}^{-1}, -K_{12})$
- (2+1)-d **GDNLS**: $(J_{21} J_{11}^{\gamma-1}, -J_{11}^{-\gamma} K_{12})$

Concluding remarks

Remark 1: The gauge factor s between DNLS equations, are usually very difficult to calculate, thus GI, CLL, KN are usually discussed separately. However, in ASDYM reduction, the gauge factor are given by $J_{11} = 1 - \theta_1 \Omega^{-1} \Xi^{-1} \eta_1$, it has a very clear expression.

Remark 2: By considering higer flows of ASDYM, the ASDYM reduction can be extended to investigate integrable hierarchies, e.g., the AKNS hierarchy, the DNLS hierarchies. [Will appear in our coming research soon...]

Remark 3: By considering GL(N) ASDYM, the ASDYM reduction can be extended to investigate matrix integrable system. [Will appear in our coming research soon...]

Concluding remarks

Question 1: The rouge wave pattern of NLS and GDNLS are investigated by bilinear method in [Yang-Yang-2021,2023], and ASDYM have rouge wave solution [Ohta-2024]. Can ASDYM rogue wave gives rise to NLS/GDNLS rogue wave?

Question 2: KP reduction is a powerful method, have been applied to investigate NLS [KIT-2001] and GDNLS [YY-2023]. What is the connection between ASDYM reduction and KP reduction?

Question 3: KP equation, CBS (Calogero-Bogoyavlenskii-Schiff) equation, DS (Davey-Stewartson) equation are also famous (2+1)-d integrable systems. How to reduce ASDYM to these equations, while the soliton structure can be preserved?

Thanks for Listening!