# From the self-dual Yang-Mills equation to the Fokas-Lenells equation

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# Outline


1 Introduction
- Self-dual Yang-Mills equation
- Reality condition
- Gauge group condition

2 Cauchy matrix approach
- Solutions of the SL(2) SDYM equation
- Solutions with asymmetric Sylvester equation
- Solutions with symmetric Sylvester equation

3 Integrable reduction
- Dimensional reduction of SDYM equation
- From SDYM equation to Fokas-Lenells equation
- Y. Matsuno's (Yamaguchi University) result

4 Conclusion and future investigations

# Introduction

# Self-dual Yang-Mills equation

The self-dual Yang-Mills (SDYM) equation [1] is an important physical model in quantum physics, gauge field theory, twistor theory and so on.

**The self-dual Yang-Mills equation**

A general form of SDYM equation is given by

$$\partial_{\tilde{z}}((\partial_z J)J^{-1}) - \partial_{\tilde{w}}((\partial_w J)J^{-1}) = 0, \qquad (1)$$

where $J$ is a matrix function of $(z, \tilde{z}, w, \tilde{w}) \in \mathbb{C}^4$.

The SDYM equation is famous in the field of integrable system since it is integrable in the sense of solving the following linear system:

$$L(\phi) \doteq (\partial_w - (\partial_w J)J^{-1})\phi - (\partial_{\tilde{z}}\phi)\zeta = 0, \qquad (2a)$$

$$M(\phi) \doteq (\partial_z - (\partial_z J)J^{-1})\phi - (\partial_{\tilde{w}}\phi)\zeta = 0. \qquad (2b)$$

[1] C.N. Yang, Condition of self-duality for SU(2) gauge fields on euclidean four-dimensional space, Phys. Rev. Lett., 38 (1977) 1377–1379.

Equation (1) is a compatible condition of (2) and usually be referred to as the $J$-matrix formulation. If we introduce $K$ that satisfies

$$\partial_{\tilde{z}} K = -(\partial_w J) J^{-1}, \qquad \partial_{\tilde{w}} K = -(\partial_z J) J^{-1}, \tag{3}$$

then, the compatible condition of (2) will also give rise to the $K$-matrix SDYM equation:

$$\partial_z \partial_{\tilde{z}} K - \partial_w \partial_{\tilde{w}} K - [\partial_{\tilde{z}} K, \partial_{\tilde{w}} K] = 0. \tag{4}$$

## Equivalence of SDYM equation in different forms

Generally speaking, equations (1), (3) and (4) can be transformed to each other. All of them can be called as the SDYM equation.


Figure 1: Equivalence of SDYM equation in different forms

# Reality condition

Since the SDYM equation is the motion equation of a certain gauge group $G$ in a certain space, their coordinates are different in various spaces.

The spaces can be classified by signature, we have Euclidean space $\mathbb{E}$, Minkowski sapce $\mathbb{M}$ and ultrahyperbolic space $\mathbb{U}$. The corresponding signatures are listed in following table.

| Space | Signature | Metric |
|---|---|---|
| $\mathbb{E}$ | $(+,+,+,+)$ | $\mathrm{d}s^2 = (\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2$ |
| $\mathbb{M}$ | $(+,-,-,-)$ | $\mathrm{d}s^2 = (\mathrm{d}x^0)^2 - (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2$ |
| $\mathbb{U}$ | $(+,+,-,-)$ | $\mathrm{d}s^2 = (\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2$ |

Table 1: Signatures and metrics of $\mathbb{E}, \mathbb{M}$ and $\mathbb{U}$ spaces

By choosing suitable coordinate transformations (See [2] for details ), for different spaces, $(z, \tilde{z}, w, \tilde{w})$ will satisfies the following reality conditions.

| Space | Reality condition |
| --- | --- |
| $\mathbb{E}$ | $\tilde{z} = z^*, \tilde{w} = -w^*$ |
| $\mathbb{M}$ | $z, \tilde{z} \in \mathbb{R}, \tilde{w} = w^*$ |
| $\mathbb{U}_1$ | $\tilde{z} = z^*,\ \tilde{w} = w^*$ |
| $\mathbb{U}_2$ | $z, \tilde{z}, w, \tilde{w} \in \mathbb{R}$ |

Table 2: Reality conditions of coordinates in various spaces

Notice that there are two coordinates to achieve the split signature $(+, +, -, -)$, and in $\mathbb{U}_2$, all of the coordinates are real. In fact, this is a important condition in integrable reduction, lots of classical soliton equations are of real coordinates. (e.g., KdV, KP, NLS and so on...)

[2] L.J. Mason, N.M.J. Woodhouse, Integrability, Self-Duality, and Twistor Theory, Oxford University Press, Oxford, New York, 1996.

# Gauge group condition

In different gauge groups (e.g. $G = \mathrm{GL}(N), \mathrm{SL}(N), \mathrm{SU}(N), \ldots$), $J$ and $K$ in (3) will satisfy certain restrictions.

| Gauge group | Gauge condition |
|---|---|
| $\mathrm{GL}(N)$ | No restrictions on $J$ and $K$ |
| $\mathrm{SL}(N)$ | $\det(J) =$ nonzero constant, $\mathrm{tr}(K) =$ constant |
| $\mathrm{U}(N)$ in $\mathbb{E}/\mathbb{U}_1$ space | $J = J^\dagger$, $J$ is positive-definite |
| $\mathrm{U}(N)$ in $\mathbb{U}_2$ space | $J \in \mathrm{SU}(N)$, $K = -K^\dagger$ |

Ablowitz et. al [3] had mentioned the $\mathrm{GL}(N)$ SDYM equation is quite important since it allows more freedom in reduction. (The Lie algebra and gauge potentials can be arbitrary) In our work, we mainly consider the case of $G = \mathrm{GL}(N)$ and $G = \mathrm{SL}(N)$.

[3] M.J. Ablowitz, S. Chakravarty, R.G. Halburd, Integrable systems and reductions of the self-dual Yang–Mills equations, J. Math. Phys., 44 (2003) 3147-3173.

# Cauchy matrix approach

# Solutions of the SL(2) SDYM equation

The Cauchy matrix approach is a systemically integrable method. It allows us to obtain explicit solutions with Cauchy matrix structure by studying the Sylvester equation.

In our recent research [4] and [5], there are progresses on constructing explicit solutions of the SDYM equation. We use the Cauchy matrix approach to derive two solution forms.

Basically, we will start from a certain Sylvester equation and certain dispersion, then construct a master function $S^{(i,j)}$, it will satisfy some relations. Finally, closed system of $S^{(i,j)}$ can be obtained by choosing special $(i, j)$.

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[4] S.S. Li, C.Z. Qu, X.X. Yi, D.J. Zhang, Cauchy matrix approach to the SU(2) self-dual Yang-Mills equation, Stud. Appl. Math., 148 (2022) 1703-1721.

[5] S.S. Li, C.Z. Qu, D.J. Zhang, Solutions to the SU($\mathcal{N}$) self-dual Yang-Mills equation, Physica D, 453 (2023) 133828 (17pp).

There are two solution forms, the difference is in the Sylvester equation:

## (i) Asymmetric Sylvester equation:

$$\boldsymbol{K}\boldsymbol{M} - \boldsymbol{M}\boldsymbol{L} = \boldsymbol{r}\boldsymbol{s}^T, \tag{5}$$

where $\boldsymbol{K}, \boldsymbol{L}, \boldsymbol{M} \in \mathbb{C}_{N\times N}$, $\boldsymbol{r} = (\boldsymbol{r}_1, \boldsymbol{r}_2) \in \mathbb{C}_{N\times 2}$, $\boldsymbol{s} = (\boldsymbol{s}_1, \boldsymbol{s}_2) \in \mathbb{C}_{N\times 2}$. By introducing $\boldsymbol{M}_1, \boldsymbol{M}_2$ that satisfy

$$\boldsymbol{K}\boldsymbol{M}_1 - \boldsymbol{M}_1\boldsymbol{L} = \boldsymbol{r}_1\boldsymbol{s}_1^T, \quad \boldsymbol{K}\boldsymbol{M}_2 - \boldsymbol{M}_2\boldsymbol{L} = \boldsymbol{r}_2\boldsymbol{s}_2^T, \tag{6}$$

we have $\boldsymbol{M} = \boldsymbol{M}_1 + \boldsymbol{M}_2$. The master functions are defined as

$$\begin{aligned}\boldsymbol{S}^{(i,j)}_{[\mathrm{asym}]} &= \boldsymbol{s}^T\boldsymbol{L}^j\boldsymbol{M}_1^{-1}\boldsymbol{K}^i\boldsymbol{r} = \boldsymbol{s}^T\boldsymbol{L}^j(\boldsymbol{M}_1+\boldsymbol{M}_2)^{-1}\boldsymbol{K}^i\boldsymbol{r} = \begin{pmatrix} s_{11}^{(i,j)} & s_{12}^{(i,j)} \\ s_{21}^{(i,j)} & s_{22}^{(i,j)} \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{s}_1^T\boldsymbol{L}^j(\boldsymbol{M}_1+\boldsymbol{M}_2)^{-1}\boldsymbol{K}^i\boldsymbol{r}_1 & \boldsymbol{s}_1^T\boldsymbol{L}^j(\boldsymbol{M}_1+\boldsymbol{M}_2)^{-1}\boldsymbol{K}^i\boldsymbol{r}_2 \\ \boldsymbol{s}_2^T\boldsymbol{L}^j(\boldsymbol{M}_1+\boldsymbol{M}_2)^{-1}\boldsymbol{K}^i\boldsymbol{r}_1 & \boldsymbol{s}_2^T\boldsymbol{L}^j(\boldsymbol{M}_1+\boldsymbol{M}_2)^{-1}\boldsymbol{K}^i\boldsymbol{r}_2 \end{pmatrix}. \end{aligned} \tag{7}$$

## (ii) Symmetric Sylvester equation:

$$\boldsymbol{K}\boldsymbol{M} - \boldsymbol{M}\boldsymbol{K} = \boldsymbol{r}\boldsymbol{s}^T, \tag{8}$$

where $\boldsymbol{K}, \boldsymbol{M}, \boldsymbol{r}, \boldsymbol{s}$ are block matrices in the forms of

$$\boldsymbol{K} = \begin{pmatrix} \boldsymbol{K}_1 & \\ & \boldsymbol{K}_2 \end{pmatrix}, \quad \boldsymbol{M} = \begin{pmatrix} & \boldsymbol{M}_1 \\ \boldsymbol{M}_2 & \end{pmatrix}, \quad \boldsymbol{r} = \begin{pmatrix} \boldsymbol{r}_1 & \\ & \boldsymbol{r}_2 \end{pmatrix}, \quad \boldsymbol{s} = \begin{pmatrix} & \boldsymbol{s}_1 \\ \boldsymbol{s}_2 & \end{pmatrix},$$

with $\boldsymbol{K}_i, \boldsymbol{M}_i \in \mathbb{C}_{N\times N}$, $\boldsymbol{r}_i, \boldsymbol{s}_i \in \mathbb{C}_{N\times 1}$, $i = 1, 2$. The master functions are defined as

$$\begin{aligned} \boldsymbol{S}^{(i,j)}_{[\mathrm{sym}]} &= \boldsymbol{s}^T\boldsymbol{K}^j(\boldsymbol{I}_{2N} + \boldsymbol{M})^{-1}\boldsymbol{K}^i\boldsymbol{r} = \begin{pmatrix} s^{(i,j)}_1 & s^{(i,j)}_2 \\ s^{(i,j)}_3 & s^{(i,j)}_4 \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{s}_2^T\boldsymbol{K}_2^j(\boldsymbol{M}_1 - \boldsymbol{M}_2^{-1})^{-1}\boldsymbol{K}_1^i\boldsymbol{r}_1 & \boldsymbol{s}_2^T\boldsymbol{K}_2^j(\boldsymbol{I}_N - \boldsymbol{M}_2\boldsymbol{M}_1)^{-1}\boldsymbol{K}_2^i\boldsymbol{r}_2 \\ \boldsymbol{s}_1^T\boldsymbol{K}_1^j(\boldsymbol{I}_N - \boldsymbol{M}_1\boldsymbol{M}_2)^{-1}\boldsymbol{K}_1^i\boldsymbol{r}_1 & \boldsymbol{s}_1^T\boldsymbol{K}_1^j(\boldsymbol{M}_2 - \boldsymbol{M}_1^{-1})^{-1}\boldsymbol{K}_2^i\boldsymbol{r}_2 \end{pmatrix}. \end{aligned}$$

# Solutions with asymmetric Sylvester equation

**Asymmetric case:** Suppose we have (See section 3 in [5]):

$$\boldsymbol{K} = \mathrm{diag}(k_1, \cdots, k_N), \quad \boldsymbol{L} = \mathrm{diag}(l_1, \cdots, l_N), \quad k_i, l_i \in \mathbb{C}, \tag{9a}$$

$$\boldsymbol{r} = (\boldsymbol{r}_1, \boldsymbol{r}_2), \quad \boldsymbol{r}_j = (\rho_j(k_1), \cdots, \rho_j(k_N))^T, \tag{9b}$$

$$\boldsymbol{s} = (\boldsymbol{s}_1, \boldsymbol{s}_2), \quad \boldsymbol{s}_j = (\sigma_j(l_1), \cdots, \sigma_j(l_N))^T, \tag{9c}$$

$$\boldsymbol{M} = \boldsymbol{M}_1 + \boldsymbol{M}_2, \quad \boldsymbol{M}_j = (M_{is}^{(j)})_{N\times N}, \quad M_{is}^{(j)} = \frac{\rho_j(k_i)\sigma_j(l_s)}{k_i - l_s}, \quad j = 1, 2, \tag{9d}$$

with the plane wave factors given by

$$\rho_j(k_i) = \exp\left(a_j \mathcal{L}(k_i) + \lambda_j(k_i)\right), \tag{10a}$$

$$\sigma_j(l_i) = \exp\left(-a_j \mathcal{L}(l_i) + \mu_j(l_i)\right), \tag{10b}$$

where

$$\mathcal{L}(k) \doteq k^n x_n + k^{n+1} x_{n+1} + k^m x_m + k^{m+1} x_{m+1}, \tag{11}$$

Then we can construct

$$\boldsymbol{V} := \boldsymbol{I}_2 - \boldsymbol{S}^{(-1,0)} = \boldsymbol{I}_2 - \boldsymbol{s}^T\boldsymbol{M}^{-1}\boldsymbol{K}^{-1}\boldsymbol{r}, \quad \boldsymbol{U} := \boldsymbol{S}^{(0,0)} = \boldsymbol{s}^T\boldsymbol{M}^{-1}\boldsymbol{r}, \tag{12}$$

they will satisfy

$$\boldsymbol{V}_{x_{n+1}}\boldsymbol{V}^{-1} = -\boldsymbol{U}_{x_n}, \quad \boldsymbol{V}_{x_{m+1}}\boldsymbol{V}^{-1} = -\boldsymbol{U}_{x_m}. \tag{13}$$

Through the following transformation, equation (13) corresponds to Miura transformation (3):

$$\boldsymbol{V} \to J, \quad \boldsymbol{U} \to K, \quad (x_{m+1}, x_n, x_{n+1}, x_m) \to (z, \tilde{z}, w, \tilde{w}). \tag{14}$$

Thus $\boldsymbol{V}$ and $\boldsymbol{U}$ solve the $J$- and $K$-matrix SDYM equation respectively.

## SL(2) gauge group condition

Through a direct calculation, we have:

$$\det(\boldsymbol{V}) = \det(\boldsymbol{L})/\det(\boldsymbol{K}), \quad \mathrm{tr}(\boldsymbol{U}) = \mathrm{tr}(\boldsymbol{K}) - \mathrm{tr}(\boldsymbol{L}). \tag{15}$$

# Solutions with symmetric Sylvester equation

**Symmetric case:** Suppose we have (See section 4 in [5]):

$$\boldsymbol{K}_1 = \mathrm{diag}(k_1, \cdots, k_N), \qquad \boldsymbol{K}_2 = \mathrm{diag}(l_1, \cdots, l_N), \tag{16a}$$

$$\boldsymbol{r}_1 = (\varrho_1(k_1), \cdots, \varrho_1(k_N))^T, \qquad \boldsymbol{s}_1 = (\varsigma_1(k_1), \cdots, \varsigma_1(k_N))^T, \tag{16b}$$

$$\boldsymbol{r}_2 = (\varrho_2(l_1), \cdots, \varrho_2(l_N))^T, \qquad \boldsymbol{s}_2 = (\varsigma_2(l_1), \cdots, \varsigma_2(l_N))^T, \tag{16c}$$

$$\boldsymbol{M}_1 = (M_{1,ij})_{N\times N}, \qquad M_{1,ij} = \frac{\varrho_1(k_i)\varsigma_2(l_j)}{k_i - l_j}, \tag{16d}$$

$$\boldsymbol{M}_2 = (M_{2,ij})_{N\times N}, \qquad M_{2,ij} = \frac{\varrho_2(l_i)\varsigma_1(k_j)}{l_i - k_j}, \tag{16e}$$

where

$$\varrho_1(k_i) = \exp\left(a_1\mathcal{L}(k_i) + \lambda_1(k_i)\right), \quad \varrho_2(l_i) = \exp\left(a_2\mathcal{L}(l_i) + \lambda_2(l_i)\right),$$
$$\varsigma_1(k_i) = \exp\left(-a_2\mathcal{L}(k_i) + \mu_1(k_i)\right), \quad \varsigma_1(k_i) = \exp\left(-a_1\mathcal{L}(k_i) + \mu_2(k_i)\right),$$

and

$$\mathcal{L}(k) \doteq k^n x_n + k^{n+1} x_{n+1} + k^m x_m + k^{m+1} x_{m+1}.$$

Then we can construct

$$\boldsymbol{V} := \boldsymbol{I}_2 - \boldsymbol{S}^{(-1,0)} = \boldsymbol{I}_2 - \boldsymbol{s}^T(\boldsymbol{I}+\boldsymbol{M})^{-1}\boldsymbol{K}^{-1}\boldsymbol{r}, \quad \boldsymbol{U} := \boldsymbol{S}^{(0,0)} = \boldsymbol{s}^T(\boldsymbol{I}+\boldsymbol{M})^{-1}\boldsymbol{r},$$

they will satisfy

$$\boldsymbol{V}_{x_{n+1}}\boldsymbol{V}^{-1} = -\boldsymbol{U}_{x_n}, \quad \boldsymbol{V}_{x_{m+1}}\boldsymbol{V}^{-1} = -\boldsymbol{U}_{x_m}. \tag{17}$$

Through the following transformation, it corresponds to (3):

$$\boldsymbol{V} \to J, \quad \boldsymbol{U} \to K, \quad (x_{m+1}, x_n, x_{n+1}, x_m) \to (z, \tilde{z}, w, \tilde{w}). \tag{18}$$

Thus $\boldsymbol{V}$ and $\boldsymbol{U}$ solve the $J$- and $K$-matrix SDYM equation respectively.

## SL(2) gauge group condition

Through a direct calculation, we have:

$$\det(\boldsymbol{V}) = 1, \quad \mathrm{tr}(\boldsymbol{U}) = 0. \tag{19}$$

# From Sylvester equation to SDYM equation

## A sketch of Cauchy matrix approach

$$\boldsymbol{K}\boldsymbol{M}-\boldsymbol{M}\boldsymbol{L}=\boldsymbol{r}\boldsymbol{s}^T$$
$$\boldsymbol{r}_{x_n}=\boldsymbol{K}^n\boldsymbol{r}\boldsymbol{a},\ \ \boldsymbol{s}_{x_n}=-(\boldsymbol{L}^T)^n\boldsymbol{s}\boldsymbol{a}$$
$$\boldsymbol{S}^{(i,j)}:=\boldsymbol{s}^T\boldsymbol{L}^j\boldsymbol{M}^{-1}\boldsymbol{K}^i\boldsymbol{r}$$

$$\boldsymbol{K}\boldsymbol{M}-\boldsymbol{M}\boldsymbol{K}=\boldsymbol{r}\boldsymbol{s}^T$$
$$\boldsymbol{r}_{x_n}=\boldsymbol{K}^n\boldsymbol{r}\boldsymbol{a},\ \ \boldsymbol{s}_{x_n}=-(\boldsymbol{K}^T)^n\boldsymbol{s}\boldsymbol{a}$$
$$\boldsymbol{S}^{(i,j)}:=\boldsymbol{s}^T\boldsymbol{L}^j(\boldsymbol{I}+\boldsymbol{M})^{-1}\boldsymbol{K}^i\boldsymbol{r}$$

$$\boldsymbol{U}:=\boldsymbol{S}^{(0,0)}$$
$$\boldsymbol{V}:=\boldsymbol{I}_2-\boldsymbol{S}^{(-1,0)}$$

$$\boldsymbol{V}_{x_{n+1}}\boldsymbol{V}^{-1}=-\boldsymbol{U}_{x_n}$$
$$\boldsymbol{V}_{x_{m+1}}\boldsymbol{V}^{-1}=-\boldsymbol{U}_{x_m}$$

$$(\boldsymbol{V}_{x_{m+1}}\boldsymbol{V}^{-1})_{x_n}-(\boldsymbol{V}_{x_{n+1}}\boldsymbol{V}^{-1})_{x_m}=0$$

$$\boldsymbol{U}_{x_n,x_{m+1}}-\boldsymbol{U}_{x_m,x_{n+1}}-[\boldsymbol{U}_{x_n},\boldsymbol{U}_{x_m}]=0$$

# Integrable reduction

# Dimensional reduction of SDYM equation

By using Cauchy matrix approach, we showed that there are two types of solution formulae for SDYM equation. Their formulae are different, but their dimensional reductions are the same.

## Dimensional reduction (i)

The $x_0$ derivative of $\boldsymbol{S}^{(i,j)}$ can be omitted:

$$\boldsymbol{S}^{(i,j)} = \boldsymbol{S}^{(i,j)}_{[\mathrm{asym}]} \text{ or } \boldsymbol{S}^{(i,j)}_{[\mathrm{sym}]}, \qquad \partial_{x_0}\boldsymbol{S}^{(i,j)} = [\boldsymbol{S}^{(i,j)}, \mathbf{a}], \tag{20a}$$

$$\boldsymbol{V}_{x_1}\boldsymbol{V}^{-1} = -[\boldsymbol{U}, \mathbf{a}], \qquad [\boldsymbol{V}, \mathbf{a}] = -\boldsymbol{U}_{x_{-1}}\boldsymbol{V}. \tag{20b}$$

## Dimensional reduction (ii)

Taking $m = n - 1$, the 4d SDYM equations reduce to 3d equations:

$$(\boldsymbol{V}_{x_n}\boldsymbol{V}^{-1})_{x_n} - (\boldsymbol{V}_{x_{n+1}}\boldsymbol{V}^{-1})_{x_{n-1}} = 0, \tag{21a}$$

$$\boldsymbol{U}_{x_n,x_n} - \boldsymbol{U}_{x_{n+1},x_{n-1}} - [\boldsymbol{U}_{x_n}, \boldsymbol{U}_{x_{n-1}}] = 0. \tag{21b}$$

By using **Dimensional reduction (i)** and **Dimensional reduction (ii)** together, we derive a 2d reduced SDYM equation.

## Reduction to sine-Gordon equation

For example, if we choose $n = 0$, $\boldsymbol{a} = \mathrm{diag}(1, -1)$, from (21b) we have:

$$[[\boldsymbol{U}, \boldsymbol{a}], \boldsymbol{a}] - \boldsymbol{U}_{x_1, x_{-1}} - [[\boldsymbol{U}, \boldsymbol{a}], \boldsymbol{U}_{x_{-1}}] = 0. \tag{22}$$

Now we define $x = x_1$, $t = x_{-1}$ and

$$\boldsymbol{U} := \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}, \tag{23}$$

then equation (22) gives rise to

$$U_{12,xt} - 4\,U_{12} + 8\,U_{12}\partial_x^{-1}(U_{12}\,U_{21})_t = 0, \tag{24a}$$

$$U_{21,xt} - 4\,U_{12} + 8\,U_{21}\partial_x^{-1}(U_{12}\,U_{21})_t = 0, \tag{24b}$$

which is the first negative member of AKNS hierarchy.

## Reduction to nonlinear Schrödinger equation

For example, if we choose $n = 1$, $\boldsymbol{a} = \mathrm{diag}(1, -1)$, from (21b) we have:

$$\boldsymbol{U}_{x_1,x_1} - [\boldsymbol{U}_{x_2}, \boldsymbol{a}] - [\boldsymbol{U}_{x_1}, [\boldsymbol{U}, \boldsymbol{a}]] = 0. \tag{25}$$

Now we define $x = x_1$, $t = -\mathrm{i}x_2$ and (23), then equation (25) gives rise to

$$\boldsymbol{U}_{xx} + \mathrm{i}[\boldsymbol{U}_t, \boldsymbol{a}] + [\boldsymbol{U}, [\boldsymbol{U}, \boldsymbol{a}]][\boldsymbol{U}, \boldsymbol{a}] = [[\boldsymbol{U}, \boldsymbol{a}], [\boldsymbol{S}^{(0,1)}, \boldsymbol{a}]]. \tag{26}$$

and

$$U_{12,xx} - 2\mathrm{i}\, U_{12,t} - 8\, U_{12}^2 U_{21} = 0, \tag{27a}$$

$$U_{21,xx} + 2\mathrm{i}\, U_{21,t} - 8\, U_{12} U_{21}^2 = 0. \tag{27b}$$

which is the second positive member of AKNS hierarchy.

# From SDYM equation to Fokas-Lenells equation

## Fokas-Lenells equation

The Fokas-Lenells equation is given by

$$u_{xt} - u - 2\mathrm{i}|u|^2 u_x = 0, \tag{28}$$

where $u = u(x,t)$ is a complex-valued function, $|u|^2 = uu^*$, $u^*$ is the conjugate of $u$, variables $x, t$ are real coordinates.

It can be regarded a reduction of the first member of negative potential Kaup-Newell hierarchy [6] [7]:

$$u_{xt} - u - 2\mathrm{i}uvu_x = 0, \tag{29a}$$

$$v_{xt} - v + 2\mathrm{i}vuv_x = 0. \tag{29b}$$

[6] J. Lenells, A.S. Fokas, On a novel integrable generalization of the nonlinear Schrödinger equation, Nonlinearity, 22 (2009) 11-27.

[7] J. Lenells, Exactly solvable model for nonlinear pulse propagation in optical fibers, Stud. Appl. Math., 123 (2009) 215-232.

Different from the AKNS hierarchy, for Fokas-Lenells equation (KN hierarchy), we are not using $U_{12}$ and $U_{21}$ to construct closed system.

## 2d reduction from Miura Trans. to pKN(-1) system

Let $m = n - 1$ and $n = 0$, assume $\det(\boldsymbol{V}) = 1$, $x = x_{-1}, t = x_1$ and $\boldsymbol{a} = \mathrm{diag}(1, 0)$, from

$$\boldsymbol{V}_{x_1}\boldsymbol{V}^{-1} = -[\boldsymbol{U}, \boldsymbol{a}], \qquad [\boldsymbol{V}, \boldsymbol{a}] = -\boldsymbol{U}_{x_{-1}}\boldsymbol{V}, \tag{30}$$

and

$$\boldsymbol{V} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}, \quad \boldsymbol{U} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}, \quad \boldsymbol{V}^{-1} = \begin{pmatrix} V_{22} & -V_{12} \\ -V_{21} & V_{11} \end{pmatrix},$$

then $u = U_{21}$ and $v = \mathrm{i}\,V_{12}/V_{22}$ solves the pKN(-1) system (29):

$$u_{xt} - u - 2\mathrm{i}uvu_x = 0,$$
$$v_{xt} - v + 2\mathrm{i}vuv_x = 0.$$

## Proof.

From (30), we have

$$\begin{pmatrix} V_{11,t} & V_{12,t} \\ V_{21,t} & V_{22,t} \end{pmatrix} = \begin{pmatrix} U_{12}V_{21} & U_{12}V_{22} \\ -U_{21}V_{11} & -U_{21}V_{12} \end{pmatrix}, \tag{31a}$$

$$\begin{pmatrix} U_{11,x} & U_{12,x} \\ U_{21,x} & U_{22,x} \end{pmatrix} = \begin{pmatrix} -V_{12}V_{21} & V_{12}V_{11} \\ -V_{21}V_{22} & V_{21}V_{12} \end{pmatrix}. \tag{31b}$$

Using the relations in (31), a direct calculation shows that

$$\begin{aligned} u_{xt} - u &= U_{21,xt} - U_{21} \\ &= (V_{11}V_{22} + V_{12}V_{21} - 1)U_{21} = 2V_{12}V_{21}U_{21} = 2\mathrm{i}vuu_x. \end{aligned}$$

This is nothing but the first equation in the pKN(−1) system (29). For the second equation (29b), taking $t$-derivative of $v$ yields

$$v_t = \mathrm{i}\left(\frac{V_{12}}{V_{22}}\right)_t = \mathrm{i}\frac{V_{12,t}V_{22} - V_{12}V_{22,t}}{V_{22}^2} = \mathrm{i}U_{12} - \mathrm{i}v^2u. \tag{32}$$

## Proof.

Furthermore, using (31), we have

$$v_{xt} = \mathrm{i}\, V_{12} V_{11} - \mathrm{i}\frac{V_{12}^2}{V_{22}} V_{21} - 2\mathrm{i}uvv_x = v - 2\mathrm{i}uvv_x, \tag{33}$$

which is (29b). Thus we complete the proof. □

This result is quite different from the AKNS case.

**The AKNS system:** We use $U_{12}$ and $U_{21}$ to construct closed system, e.g. sine-Gordon equation...

**The KN system:** We use $U_{21}$ and $\mathrm{i}\, V_{12}/V_{22}$ to construct closed system, e.g. Fokas-Lenells equation...

However, we can see, both of the AKNS(-1) system $(U_{12}, U_{21})$ and pKN(-1) system $(U_{21}, \mathrm{i}\, V_{12}/V_{22})$, can be derived from

$$\boldsymbol{V}_{x_1} \boldsymbol{V}^{-1} = -[\boldsymbol{U}, \boldsymbol{a}], \quad [\boldsymbol{V}, \boldsymbol{a}] = -\boldsymbol{U}_{x_{-1}} \boldsymbol{V}.$$

# Conjugate reduction

Now we have proved that $u = U_{21}$ and $v = \mathrm{i}\, V_{12}/V_{22}$ solve the pKN(-1) system. To achieve the FL equation, we need to find certain restriction to make $u = v^*$, i.e. $U_{21}^\dagger V_{22} = \mathrm{i}\, V_{12}$.

## Reduction for asymmetric case:

If we use the Cauchy matrix approach with asymmetric Sylvester equation, the reduction is given by:

$$\boldsymbol{L} = -\boldsymbol{K}^\dagger, \quad \boldsymbol{s}_1^T = -\mathrm{i}\boldsymbol{r}_1^\dagger \boldsymbol{K}^\dagger, \quad \boldsymbol{s}_2^T = \boldsymbol{r}_2^\dagger. \tag{34}$$

## Reduction for symmetric case:

If we use the Cauchy matrix approach with symmetric Sylvester equation, the reduction is given by:

$$\boldsymbol{K}_2 = -\boldsymbol{K}_1^\dagger, \quad \boldsymbol{s}_2 = \boldsymbol{r}_1^*, \quad \boldsymbol{r}_2 = \mathrm{i}\boldsymbol{K}_1^* \boldsymbol{s}_1^*. \tag{35}$$

## N-soliton solution of the FL equation (asymmetric-type)

Suppose

$$\boldsymbol{K} = \mathrm{diag}(k_1, \cdots, k_N), \qquad \boldsymbol{r}_j = (\rho_j(k_1), \cdots, \rho_j(k_N))^T, \qquad j = 1, 2, \tag{36a}$$

$$\boldsymbol{M} = (M_{ij})_{N\times N}, \qquad M_{ij} = \frac{\rho_2(k_i)(\rho_2(k_j))^* - \mathrm{i}\rho_1(k_i)(\rho_1(k_j))^* k_j^*}{k_i + k_j^*}, \tag{36b}$$

where

$$\rho_1(k_i) = \exp\left(\frac{1}{k_i}x + k_i t + \lambda_1(k_i)\right), \qquad \rho_2(k_i) = \exp\left(\lambda_2(k_i)\right),. \tag{37}$$

Then the following function

$$u_{[\mathrm{asym}]} = \boldsymbol{r}_2^{\dagger}\boldsymbol{M}^{-1}\boldsymbol{r}_1 \tag{38}$$

provides a $N$-soliton solution of the FL equation (28).

# N-soliton solution of the FL equation (symmetric-type)

$$u_{[\mathrm{sym}]} = \boldsymbol{s}_1^T (\boldsymbol{I}_N - \boldsymbol{M}_1 \boldsymbol{M}_2)^{-1} \boldsymbol{r}_1 \tag{39}$$

gives a $N$-soliton solution of the FL equation (28), where

$$\boldsymbol{K} = \mathrm{diag}(k_1, \cdots, k_N), \tag{40a}$$

$$\boldsymbol{r}_1 = (\varrho_1(k_1), \cdots, \varrho_1(k_N))^T, \qquad \boldsymbol{s}_1 = (\varsigma_1(k_1), \cdots, \varsigma_1(k_N))^T, \tag{40b}$$

$$\boldsymbol{M}_1 = (M_{1,ij})_{N\times N}, \qquad M_{1,ij} = \frac{\varrho_1(k_i)(\varrho_1(k_j))^*}{k_i + k_j^*}, \tag{40c}$$

$$\boldsymbol{M}_2 = (M_{2,ij})_{N\times N}, \qquad M_{2,ij} = -\frac{\mathrm{i} k_i^* (\varsigma_1(k_i))^* \varsigma_1(k_j)}{k_i^* + k_j}, \tag{40d}$$

the plane wave factors are give by

$$\varrho_1(k_i) = \exp\left(\frac{1}{k_i} x + k_i t + \lambda_1(k_i)\right), \qquad \varsigma_1(k_i) = \exp\left(\mu_1(k_i)\right), \tag{41}$$

# Y. Matsuno's (Yamaguchi University) result

If anybody is familiar with Y. Matsuno's work [8] and [9] in 2012, he must have some questions about the difference between our CMA solution (asym-type and sym-type) with the Matsuno solution.

In fact, Matsuno had used bilinear method to obtain the $N$-soliton formula of FL equation:

$$\mathrm{u}_{xt} - \mathrm{u} + 2\mathrm{i}|\mathrm{u}|^2\mathrm{u}_x = 0. \tag{42}$$

By introducing bilinear transformation $\mathrm{u} = g/f$, we can rewrite (42) as

$$D_xD_tg \cdot f = gf, \quad D_tf \cdot f^* = \mathrm{i}gg^*, \quad D_xD_tf \cdot f^* = \mathrm{i}D_xg \cdot g^*. \tag{43}$$

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[8] Y. Matsuno, A direct method of solution for the Fokas-Lenells derivative nonlinear Schrödinger equation: I. Bright soliton solutions, J. Phys. A: Math. Theor., 45 (2012) 235202 (19pp).

[9] Y. Matsuno, A direct method of solution for the Fokas-Lenells derivative nonlinear Schrödinger equation: II. Dark soliton solutions, J. Phys. A: Math. Theor., 45 (2012) 475202 (31pp).

Then, by Theorem 3.1, the following constructions give rise to the following bright $N$-soliton solution of the FL equation:

$$f = |W|, \quad g = \begin{vmatrix} W & \mathbf{z}_t^T \\ \mathbf{1} & 0 \end{vmatrix}, \quad W = (d_{ij})_{N\times N}, \quad d_{ij} = \frac{z_i z_j^* - \mathrm{i} p_j^*}{p_i + p_j*},$$

$$z_i = \exp\left(p_i x + \frac{1}{p_i} t + \zeta_{i0}\right), \quad \mathbf{z}_t = (z_1/p_1, \cdots, z_N/p_N), \quad \mathbf{1} = (1, \cdots, 1).$$

## The correspondence to Matsuno solution

Consider the transformations:

$$p_i \to -\frac{1}{k_i}, \quad z_i \to \exp\left(-\frac{1}{k_i} x - k_i t + \zeta_{i0}\right), \quad \mathbf{z}_t \to (-k_1 z_1, \cdots, -k_N z_N),$$

$$\rho_2(k_i) \to k_i z_i, \qquad \rho_1(k_i) \to 1$$

Then we have $W = -\boldsymbol{M}^\dagger$, $\mathbf{1} = \boldsymbol{r}_1^\dagger$ and $\mathbf{z}_t^T = -\boldsymbol{r}_2$. Thus we can rewrite Matsuno's solution as

$$\mathrm{u} = -\mathbf{1}\, W^{-1} \mathbf{z}_t^T = -\boldsymbol{r}_1^\dagger \boldsymbol{M}^{-\dagger} \boldsymbol{r}_2 = -(\boldsymbol{r}_2^\dagger \boldsymbol{M}^{-1} \boldsymbol{r}_1)^\dagger = -u^*_{[\mathrm{asym}]}. \tag{45}$$

Matsuno also indicated that there is an alternative expression $\mathrm{u} = g'/f'$ that also gives rise to $N$-soliton solution formula:

$$f' = \begin{vmatrix} A & I \\ -I & B \end{vmatrix}, \qquad g' = \begin{vmatrix} A & I & \mathbf{y}_t^T \\ -I & B & \mathbf{0}^T \\ \mathbf{0} & \mathbf{1} & 0 \end{vmatrix}, \tag{46}$$

where

$$A = (a_{ij})_{N\times N}, \qquad a_{ij} = \frac{y_i y_j^*}{q_i + q_j^*}, \qquad y_i = \exp\left(q_i x + \frac{1}{q_i} t + \eta_{i0}\right), \tag{47a}$$

$$B = (b_{ij})_{N\times N}, \qquad b_{ij} = \frac{\mathrm{i} q_j}{q_i^* + q_j}, \qquad \mathbf{y}_t = (y_1/q_1, \cdots, y_N/q_N). \tag{47b}$$

# The correspondence to Matsuno solution

Consider the transformations:

$$q_i \to \frac{1}{k_i}, \quad y_i \to \exp\left(\frac{1}{k_i}x + k_i t + \eta_{i0}\right), \quad \mathbf{y}_t = (k_1 y_1, \cdots, k_N y_N),$$
$$\rho_1(k_i) \to k_i y_i, \quad \sigma_1(k_i) \to 1$$

which implies $A = \boldsymbol{M}_1$, $B = -\boldsymbol{M}_2$, $\mathbf{y}_t = \boldsymbol{r}_1^T$, $\mathbf{1} = \boldsymbol{s}_1^T$. Then we have

$$f' = |AB + I| = |\boldsymbol{I}_N - \boldsymbol{M}_1\boldsymbol{M}_2|, \tag{48}$$

$$g' = \begin{vmatrix} I + AB & \mathbf{y}_t^T \\ \mathbf{1} & 0 \end{vmatrix} = \begin{vmatrix} \boldsymbol{I}_N - \boldsymbol{M}_1\boldsymbol{M}_2 & \boldsymbol{r}_1 \\ \boldsymbol{s}_1^T & 0 \end{vmatrix}, \tag{49}$$

which shows

$$\mathrm{u} = \frac{g'}{f'} = -\boldsymbol{s}_1^T(\boldsymbol{I}_N - \boldsymbol{M}_1\boldsymbol{M}_2)^{-1}\boldsymbol{r}_1 = -u_{[\mathrm{sym}]}, \tag{50}$$

# Conclusion and future investigations

# Conclusion and future investigations

**What we have done in this paper:**

**1.** This paper is a continuous work of our previous researches on using Cauchy matrix approach to derive solutions of the SDYM equation. There are two types of them: (i) Asymmetric case (ii) Symmetric case.

**2.** Starting from SDYM equation, under our Cauchy matrix frame, we derive the following 2d system (Reduced Miura transformation):

$$\boldsymbol{V}_{x_1}\boldsymbol{V}^{-1} = -[\boldsymbol{U}, \boldsymbol{a}], \qquad [\boldsymbol{V}, \boldsymbol{a}] = -\boldsymbol{U}_{x_{-1}}\boldsymbol{V}.$$

We use $u = U_{21}$ and $v = \mathrm{i}\, V_{12}/V_{22}$ to construct the pKN(-1) system.

**3.** We assume some conjugate reduction so that we have $v = u^*$ and pKN(-1) system goes to FL equation. Finally, we have derived two kinds of solution forms (Asym-type and Sym-type).

## SDYM reduction by using Lax pair

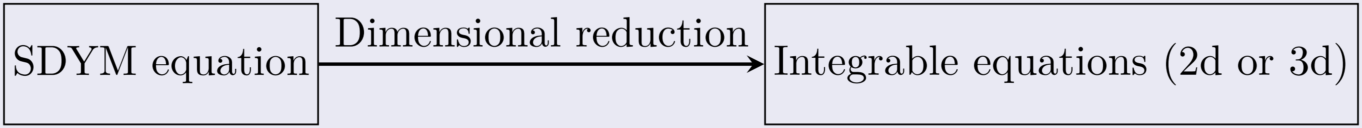


Figure 2: Ablowitz et. al's work

## SDYM reduction by using Cauchy matrix approach

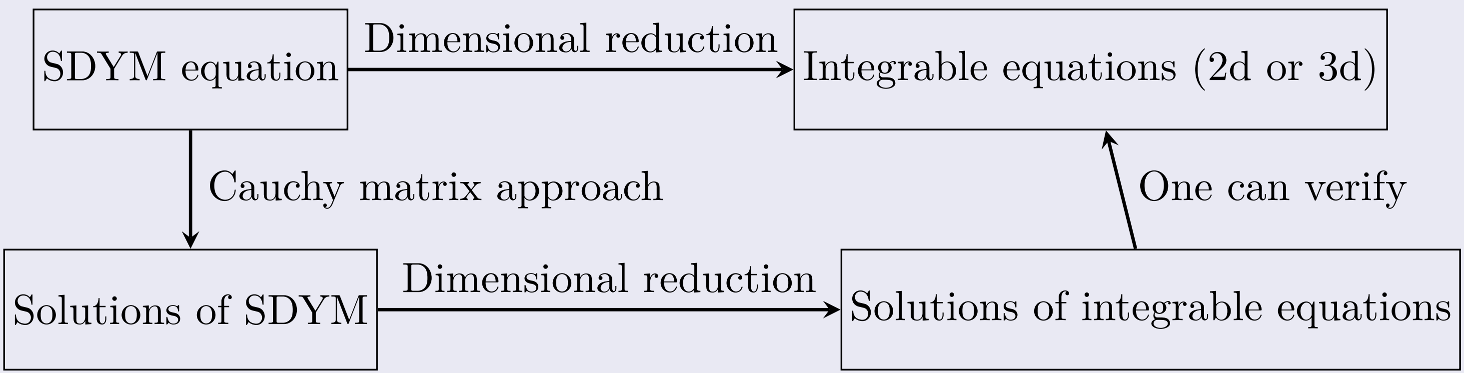


Figure 3: SDYM reduction under CMA framework

We have proved that our two types of solutions corresponds to the Matsuno solution (two types) respectively.

**Different points: 1.** We investigate the pKN(-1) system first, then reduce it to the FL equation.

$$\boxed{\begin{aligned} &u_{xt} - u - 2\mathrm{i}uvu_x = 0,\\ &v_{xt} - v + 2\mathrm{i}vuv_x = 0.\end{aligned}} \xrightarrow{v = u^*} \boxed{u_{xt} - u - 2\mathrm{i}|u|^2 u_x = 0,}$$

**2.** We investigate the nonlinear variable $(u, v)$, rather than the tau function $(f, g)$, so we don't have to calculate the bilinear derivatives.

**3. (My personal comment...)** Matsuno sensei's construction is too magic. In his paper, he gave the formula of $f$ and $g$ directly, then he prove that $u = g/f$ solves the FL equation.

**4.** We study the FL equation from the perspective of dimensional reduction of SDYM equation, it is more reasonable.

For sine-Gordon equation (in AKNS hierarchy) and Fokas-Lenells equation (in KN hierarchy), we use the following reduced Miura transformation:

$$\boldsymbol{V}_{x_1}\boldsymbol{V}^{-1} = -[\boldsymbol{U}, \boldsymbol{a}], \qquad [\boldsymbol{V}, \boldsymbol{a}] = -\boldsymbol{U}_{x_{-1}}\boldsymbol{V},$$

where we choose (for sine-Gordon case):

$$\boldsymbol{V} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}, \quad \boldsymbol{U} = \begin{pmatrix} U_{11} & \boxed{U_{12}} \\ \boxed{U_{21}} & U_{22} \end{pmatrix},$$

and we choose (for Fokas-Lenells case):

$$\boldsymbol{V} = \begin{pmatrix} V_{11} & \boxed{V_{12}} \\ V_{21} & \boxed{V_{22}} \end{pmatrix}, \quad \boldsymbol{U} = \begin{pmatrix} U_{11} & U_{12} \\ \boxed{U_{21}} & U_{22} \end{pmatrix},$$

In this matrix equation system, we can see both AKNS and KN in it!

# Ward's conjecture

There is a well-known conjecture proposed by Ward [10]:

*... many (and perhaps all?) of the ordinary and partial differential equations that are regarded as being integrable or solvable may be obtained from the self-dual gauge field equations (or its generalizations) by reduction.*

Many researchers have contributed a lot to verify this conjecture, lot of classical integrable equations are shown to be some kinds of integrable reduction of the SDYM equation. Our work provides a new example about it.

Consider about the Cauchy matrix approach, it is also a powerful tool to implement integrable discretization, e.g. [11] May be one can consider discretization of some equations in the Kaup-Newell hierarchy [12].

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[11] A.A. Cho, J. Wang, D.J. Zhang, Discretization of the modified Korteweg-de Vries-sine Gordon equation, Theore. Math. Phys., 217 (2023) 1700-1716

[12] F.W.Nijhoff, H.W. Capel, G.R.W. Quispel, Integrable lattice version of the massive Thirring model and its linearization, Phys. Lett. A, 98 (1983) 83-86.

# Thanks for Listening!