コーシー行列法、シルベスター方程式、自己双対ヤンミルズ方程式、およびそれらの可積分系との関係

講演者: 李上帅
所属: 上海大学と早稲田大学
時間: 2024/11/08
場所: 福井大学
会議: 非線形波動から可積分系へ2024

Cauchy matrix approach, Sylvester equation, self-dual Yang-Mills equation and their relationship with integrable system

Speaker: Shangshuai Li

Affiliation: Shanghai University & Waseda University

Date: 2024/11/08

Place: University of Fukui

Conference: 非線形波動から可積分系へ2024

Outline

- **Background of Cauchy matrix**
  - -What is Cauchy matrix? The properties and formula.
- **Cauchy matrix approach: KP equation**
  - -How does Cauchy matrix approach work? Use KP equation as an example
- **Self-dual Yang-Mills equation**
  - -Why self-dual Yang-Mills equation is so important in integrable systems?
  - -Ward conjecture
- **Cauchy matrix structure of self-dual Yang-Mills equation**
  - -How to construct SDYM Cauchy matrix solution
  - -Integrable reduction towards 2d/3d integrable systems

Background of Cauchy matrix

Background of Cauchy matrix

- The Cauchy matrix is one kind of the matrix in the form of:

$$\boldsymbol{M} = (m_{ij})_{N\times M}, \quad m_{ij} = \frac{1}{k_i - l_j}, \quad 1 \le i \le N, \quad 1 \le j \le M, \quad k_i \ne k_j.$$

- It can be regarded as a solution of the following Sylvester equation:

$$\boldsymbol{KM} - \boldsymbol{ML} = \boldsymbol{C}, \quad \boldsymbol{K} = \mathrm{diag}(k_1, \cdots, k_N), \quad \boldsymbol{L} = \mathrm{diag}(l_1, \cdots, l_M), \quad \boldsymbol{C} = (c_{ij})_{N\times M}, \quad c_{ij} = 1.$$

- Cauchy matrix is invertible, the inverse matrix is still a Cauchy matrix, there is a determinant formula for Cauchy matrix (see more details in [1]).

[1] https://en.wikipedia.org/wiki/Cauchy_matrix

Appearance of Cauchy matrix in integrable system

- Cauchy matrix is a famous matrix in computer science, control theory and linear algebra. But it also plays an important role in integrable system, this is the main thema of our talk.
- To author's best knowledge, it was first appeared in 1956 [2] to solve the one-dimensional Schrödinger equation, there are some formulae:

$$\hat{A} = \left\| A_n^{\frac{1}{2}} A_\nu^{\frac{1}{2}} \exp[(\kappa_n+\kappa_\nu)x] / (\kappa_n+\kappa_\nu) \right\| = DAD^{-1}. \quad (2.6)$$

→ Cauchy matrix with ($l_j = -k_j$)

$$u(x,E) = \{1 - A_1 \exp(2\kappa_1 x) / [1 + A_1 \exp(2\kappa_1 x)/2\kappa_1] \times [\kappa_1 + i\sqrt{E}]\} \exp(i\sqrt{E}x), \quad (4.3)$$

→ 1-soliton tau function

[2] I. Kay, H.E. Moses, Reflectionless Transmission through Dielectrics and Scattering Potentials, J. Appl. Phys. 27 (1956) 1503-1508.

Appearance of Cauchy matrix in integrable system

- Cauchy matrix in KP equation [3,4]:

$$(-4u_t + u_{xxx} + 6uu_x)_x + 3\beta u_{yy} = 0,$$

KP-II equation (beta=1)

$$u(x, y, t) = 2(\ln \tau)_{xx},$$

Bilinear transformation

$$\tau = \det(\Omega), \quad \Omega_{ij} = \int_\Gamma f_i g_j \,\mathrm{d}x - (f_i g_{jx} - f_{ix} g_j)\mathrm{d}y, \quad i, j = 1, 2, \ldots, N,$$

Grammian solution

$$\Omega_{ij}(x, y, t) = c_{ij} + \int_\Gamma f_i g_j \,\mathrm{d}x = c_{ij} + \sum_{m=1}^{r} \sum_{n=1}^{N} b_{im} \widehat{b}_{jn} \frac{e^{\phi(p_m) - \phi(q_n)}}{p_m - q_n},$$

Cauchy matrix structure

[3] S. Chakravarty, T. Lewkow, K.I. Maruno, On the construction of the KP line-solitons and their interactions, Appl. Anal., 89 (2010) 529-545.
[4] V.B. Matveev, M.A. Salle, Darboux Transformations and Solitons, Springer-Verlag, Berlin, 1991.

Appearance of Cauchy matrix in integrable system

- Cauchy matrix in Toda equation [5]:

$$\frac{d^2 x_n}{dt^2} = \exp(x_{n-1} - x_n) - \exp(x_n - x_{n+1})$$

Toda lattice

$$x_n = \tilde{P}t + \log\det(B(n)B^{-1}(n-1)),$$

$$k_i = \exp(-\gamma_i), \quad l_j = \exp(\gamma_j)$$

Solution formula

$$\delta_{ij} + \frac{c_i(0)c_j(0)\exp((\sinh\gamma_i + \sinh\gamma_j)t - (\gamma_i + \gamma_j)(n+1))}{1 - \exp(\gamma_i + \gamma_j)},$$

Tau function with Cauchy matrix

[5] T. Makato, The solution of Toda equation and dimensional functions, 数理解析研究所講究録, 933 (1995) 35-43.

[6] https://en.wikipedia.org/wiki/Toda_lattice

Appearance of Cauchy matrix in integrable system

- A comprehensive literature [7] had collected soliton equations that the solution has Cauchy matrix structure, including: KP equation, sine-Gordon equation, NLS equation, Toda chain and 2D Toda lattice.

- There are lots of integrable systems have the solution of the Cauchy matrix form. And sometimes they are referred to as the Grammian solution, a complete different name.

[7] C. Schiebold, Cauchy-type determinants and integrable systems, Linear Algebra Appl., 433 (2010) 447–475.

Table 1
Inverse images of elementary operators in explicit solution formulas for integrable systems ($\Phi_{A,B}X := AX + XB$ and $\Psi_{A,B}X := AXB - X$).

Sine-Gordon equation	$\Phi_{A,A}^{-1}(a \otimes c)$
Kadomtsev–Petviashvili equation	$\Phi_{A,B}^{-1}(a \otimes c)$
Matrix Kadomtsev–Petviashvili equation	$C = \Phi_{A,B}^{-1}\left(\sum_{k=1}^{N} a_k \otimes c_k\right)$ and $C + a_i \otimes c_j \ \forall i, j$
Nonlinear Schrödinger equation	$\begin{pmatrix} 0 & \Phi_{A,B}^{-1}(b \otimes c) \\ \Phi_{B,A}^{-1}(a \otimes d) & 0 \end{pmatrix}$, $\begin{pmatrix} -a \otimes c & \Phi_{A,B}^{-1}(b \otimes c) \\ \Phi_{B,A}^{-1}(a \otimes d) & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & \Phi_{A,B}^{-1}(b \otimes c) \\ \Phi_{B,A}^{-1}(a \otimes d) & -b \otimes d \end{pmatrix}$
Toda lattice	$\Psi_{A,A}^{-1}(a \otimes c)$
2-Dimensional Toda lattice	$\Psi_{A,B}^{-1}(a \otimes c)$

[7] C. Schiebold, Cauchy-type determinants and integrable systems, Linear Algebra Appl., 433 (2010) 447–475.

Cauchy matrix and Gram matrix

- Usually, the Cauchy matrix in tau function is equivalent to the Gram matrix.
- Cauchy matrix: $\boldsymbol{M} = (m_{ij})_{N\times M}, \quad m_{ij} = \frac{\rho(k_i)\sigma(l_j)}{k_i - l_j},$
- Gram matrix: $\boldsymbol{G} = \int \boldsymbol{r}\boldsymbol{s}^T \mathrm{d}x, \quad \boldsymbol{r} = (\rho(k_1), \cdots, \rho(k_N))^T, \quad \boldsymbol{s} = (\sigma(l_1), \cdots, \sigma(l_M))^T$
- Plane wave factors: $\rho(k_i) = \exp(k_i x + \cdots), \quad \sigma(l_j) = \exp(-l_j x + \cdots).$
- Equivalence: $\int \rho(k_i)\sigma(l_j)\mathrm{d}x = \int \exp((k_i - l_j)x + \cdots)\mathrm{d}x = \frac{\rho(k_i)\sigma(l_j)}{k_i - l_j}$

Cauchy matrix approach: KP equation

Cauchy matrix approach: KP equation

- In this part, we will introduce how the Cauchy matrix approach works and how to derive soliton solutions with Cauchy matrix structure. One can refer to [8] for calculation details.

The potential KP (pKP) equation is given by

$$u_t - u_{xxx} - 6u_x^2 - \partial^{-1}u_{yy} = 0, \tag{1.1}$$

where $\partial^{-1} = \partial_x^{-1} = \int \mathrm{d}x$. Then by transformation $\omega = 2u_x$, it becomes the KP equation

$$(\omega_t - \omega_{xxx} - 6\omega\omega_x)_x - 3\omega_{yy} = 0. \tag{1.2}$$

[8] W. Feng, S.L. Zhao, The Sylvester Equation and Kadomtsev–Petviashvili System, Symmetry, 14 (2022) 542 (17pp).

Cauchy matrix approach: KP equation

To construct the Cauchy matrix solution of the pKP equation, we introduce the following Cauchy matrix scheme, which contains a Sylvester equation and three dispersion relationships:

$$\boldsymbol{K}\boldsymbol{M} - \boldsymbol{M}\boldsymbol{L} = \boldsymbol{r}\boldsymbol{s}^T, \tag{1.3a}$$

$$\boldsymbol{r}_x = \boldsymbol{K}\boldsymbol{r}, \quad \boldsymbol{s}_x = -\boldsymbol{L}^T\boldsymbol{s}, \tag{1.3b}$$

$$\boldsymbol{r}_y = -\boldsymbol{K}^2\boldsymbol{r}, \quad \boldsymbol{s}_y = (\boldsymbol{L}^T)^2\boldsymbol{s}, \tag{1.3c}$$

$$\boldsymbol{r}_t = 4\boldsymbol{K}^3\boldsymbol{r}, \quad \boldsymbol{s}_t = -4(\boldsymbol{L}^T)^3\boldsymbol{s}. \tag{1.3d}$$

It follows the construction in [7] with $\boldsymbol{K} \in \mathbb{C}_{N\times N}$, $\boldsymbol{L} \in \mathbb{C}_{N'\times N'}$, $\boldsymbol{M}(x,y,t) \in \mathbb{C}_{N\times N'}$, $\boldsymbol{r}(x,y,t) \in \mathbb{C}_{N\times 1}$ and $\boldsymbol{s}(x,y,t) \in \mathbb{C}_{N'\times 1}$. Then we introduce a master function determined by

$$S^{(i,j)} = \boldsymbol{s}^T\boldsymbol{L}^j\boldsymbol{C}(\boldsymbol{I}_N + \boldsymbol{M}\boldsymbol{C})^{-1}\boldsymbol{K}^i\boldsymbol{r}, \tag{1.4}$$

where $\boldsymbol{C} \in \mathbb{C}_{N'\times N}$ is a constant matrix.

- By using (1.3), the derivatives of $S^{(i,j)}$ can be calculated:

$$S_x^{(i,j)} = S^{(i+1,j)} - S^{(i,j+1)} - S^{(i,0)}S^{(0,j)},$$
$$S_y^{(i,j)} = -S^{(i+2,j)} + S^{(i,j+2)} + S^{(i,1)}S^{(0,j)} + S^{(i,0)}S^{(1,j)},$$
$$S_t^{(i,j)} = 4(S^{(i+3,j)} - S^{(i,j+3)} - S^{(i,0)}S^{(2,j)} - S^{(i,1)}S^{(1,j)} - S^{(i,2)}S^{(0,j)}).$$

$$\begin{aligned} S_{xxx}^{(i,j)} =& S^{(i+3,j)} - S^{(i,j+3)} - 3S^{(i+2,j+1)} + 3S^{(i+1,j+2)} - 3S^{(i+2,0)}S^{(0,j)} \\ & - 3S^{(i,0)}S^{(0,j+2)} + 6S^{(i+1,0)}S^{(0,j+1)} - 3S^{(i+1,0)}S^{(1,j)} - 3S^{(i,1)}S^{(0,j+1)} \\ & - S^{(i,2)}S^{(0,j)} - S^{(i,0)}S^{(2,j)} + 3S^{(i+1,1)}S^{(0,j)} + 3S^{(i,0)}S^{(1,j+1)} \\ & + 6S^{(i+1,0)}S^{(0,0)}S^{(0,j)} - 6S^{(i,0)}S^{(0,0)}S^{(0,j+1)} + 2S^{(i,1)}S^{(1,j)} \\ & + 3S^{(i,0)}S^{(0,0)}S^{(1,j)} + 3S^{(i,0)}S^{(1,0)}S^{(0,j)} - 3S^{(i,0)}S^{(0,1)}S^{(0,j)} \\ & - 3S^{(i,1)}S^{(0,0)}S^{(0,j)} - 6S^{(i,0)}{S^{(0,0)}}^2 S^{(0,j)}, \end{aligned}$$

$$\begin{aligned} \partial^{-1}S_{yy}^{(i,j)} =& S^{(i+3,j)} - S^{(i,j+3)} + S^{(i+2,j+1)} - S^{(i+1,j+2)} - S^{(0,j)}S^{(i+1,1)} \\ & - S^{(0,j+1)}S^{(i,1)} - S^{(1,j)}S^{(i+1,0)} - S^{(1,j+1)}S^{(i,0)} \\ & + \partial_y\partial^{-1}(S^{(i,0)}S_x^{(0,j)} - S_x^{(i,0)}S^{(0,j)}), \end{aligned}$$

Cauchy matrix approach: KP equation

In section 3.3 of [7] the authors proved that $u = S^{(0,0)}$ solves the pKP equation, thus $\omega = 2(S^{(0,0)})_x$ solves the KP equation. In terms of the terminology of quasideterminant, one can rewrite u as

$$u = -\begin{vmatrix} \boldsymbol{I}_N + \boldsymbol{M}\boldsymbol{C} & \boldsymbol{r} \\ \boldsymbol{s}^T\boldsymbol{C} & \boxed{0} \end{vmatrix} = -\frac{\begin{vmatrix} \boldsymbol{I}_N + \boldsymbol{M}\boldsymbol{C} & \boldsymbol{r} \\ \boldsymbol{s}^T\boldsymbol{C} & 0 \end{vmatrix}}{|\boldsymbol{I}_N + \boldsymbol{M}\boldsymbol{C}|}. \tag{1.5}$$

We can prove that

Bilinear transformation

$$u = \mathrm{Tr}(\boldsymbol{s}^T\boldsymbol{C}(\boldsymbol{I}_N + \boldsymbol{M}\boldsymbol{C})^{-1}\boldsymbol{r}) = \frac{|\boldsymbol{I}_N + \boldsymbol{M}\boldsymbol{C}|_x}{|\boldsymbol{I}_N + \boldsymbol{M}\boldsymbol{C}|}. \tag{1.6}$$

In this case $\tau = |\boldsymbol{I}_N + \boldsymbol{M}\boldsymbol{C}|$ serves as the bilinear variable of the bilinear KP equation.

Theorem 1. *Suppose*

$$\boldsymbol{K} = \mathrm{diag}(k_1, \cdots, k_N), \qquad \boldsymbol{L} = \mathrm{diag}(l_1, \cdots, l_N), \tag{1.7}$$

$$\boldsymbol{r} = (\rho_1, \cdots, \rho_N)^T, \qquad \boldsymbol{s} = (\sigma_1, \cdots, \sigma_N)^T, \tag{1.8}$$

$$\boldsymbol{M} = (m_{ij})_{1\leq i.j \leq N}, \qquad m_{ij} = \frac{\rho_i \sigma_j}{k_i - l_j}, \qquad \boldsymbol{C} = \boldsymbol{I}_N, \tag{1.9}$$

where

$$\rho_i = \rho(k_i) = \exp(k_i x - k_i^2 t + 4k_i^3 + \lambda_i), \qquad \sigma_j = \sigma(l_j) = \exp(-l_j x + l_j^2 y - 4l_j^3 t + \mu_j). \tag{1.10}$$

For convenience, we define linear functions:

$$\theta_i = \mathcal{L}(k_i) = k_i x - k_i^2 t + 4k_i^3, \qquad \eta_j = \mathcal{L}(l_j) = l_j x - l_j^2 t + 4l_j^3, \tag{1.11}$$

and rewrite ρ_i, σ_j as

$$\rho_i = a_i \mathbf{e}^{\theta_i}, \qquad \sigma_j = b_j \mathbf{e}^{-\eta_j}. \tag{1.12}$$

Then $\omega = 2(\boldsymbol{s}^T(\boldsymbol{I} + \boldsymbol{M})^{-1}\boldsymbol{r})_x$ give rise to N-soliton solution of the KP equation.

Cauchy matrix approach: KP equation

- We start from a Sylvester equation and dispersion relationships.
- Then we determine a master function $S^{(i,j)}$
- The derivatives can be calculated
- We apply these relations to derive (KP) equations of $u=S^{(0,0)}$

$$\boldsymbol{K}\boldsymbol{M} - \boldsymbol{M}\boldsymbol{L} = \boldsymbol{r}\boldsymbol{s}^T,$$
$$\boldsymbol{r}_x = \boldsymbol{K}\boldsymbol{r}, \quad \boldsymbol{s}_x = -\boldsymbol{L}^T\boldsymbol{s},$$
$$\boldsymbol{r}_y = -\boldsymbol{K}^2\boldsymbol{r}, \quad \boldsymbol{s}_y = (\boldsymbol{L}^T)^2\boldsymbol{s},$$
$$\boldsymbol{r}_t = 4\boldsymbol{K}^3\boldsymbol{r}, \quad \boldsymbol{s}_t = -4(\boldsymbol{L}^T)^3\boldsymbol{s}.$$

$$S^{(i,j)} = \boldsymbol{s}^T\boldsymbol{L}^j\boldsymbol{C}(\boldsymbol{I}_N + \boldsymbol{M}\boldsymbol{C})^{-1}\boldsymbol{K}^i\boldsymbol{r},$$

$$u = S^{(0,0)}$$

$$u_t - u_{xxx} - 6u_x^2 - \partial^{-1}u_{yy} = 0,$$

How to construct SDYM Cauchy matrix solution?

- If we want to construct SDYM equation by using Cauchy matrix approach, we have to find certain Sylvester equation and dispersion relationships.

- The SDYM equation is a matrix equation, so we have to construct the master function as a matrix function.

- SDYM equation has lots of reductions to classical integrable systems, can SDYM Cauchy matrix solution generate Cauchy matrix solutions of other equations? (Ward conjecture)

Self-dual Yang-Mills equation

Self-dual Yang-Mills equation

- The self-dual Yang-Mills equation is an important physics model in quantum physics, geometry (fiber bundles), integrable system.

- Self-dual is a concept from gauge field theory, it means the field strength equals to its duality, i.e. :

$$F_{\mu\nu} = *F_{\mu\nu} := \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}.$$

Self-dual

Definition of dualilty strength

Self-dual Yang-Mills equation

- The orginal self-dual condition is difficult to deal with. In 1977[9], Yang had introduced a coordinate transformation and rewrite the self-dual condition as the follows:

$$F_{yz} = F_{\bar{y}\bar{z}} = 0, \qquad F_{y\bar{y}} + F_{z\bar{z}} = 0.$$

Equivalent form of self-dual condition

$$f(f_{y\bar{y}} + f_{z\bar{z}}) - f_y f_{\bar{y}} - f_z f_{\bar{z}} - e_y g_{\bar{y}} - e_z g_{\bar{z}} = 0,$$
$$f(e_{y\bar{y}} + e_{z\bar{z}}) - 2e_y f_{\bar{y}} - 2e_z f_{\bar{z}} = 0,$$
$$f(g_{y\bar{y}} + g_{z\bar{z}}) - 2g_{\bar{y}} f_y - 2g_{\bar{z}} f_z = 0,$$

R-gauge equations

[9] C.N. Yang, Condition of Self-Duality for SU(2) Gauge Fields on Euclidean Four-Dimensional Space, Phys. Rev. Lett., 38 (1977) 1377-1379.

Self-dual Yang-Mills equation

- From R-gauge equations, through the following construction, a matrix equation can be derived[10,11]:

$$J = \frac{1}{f}\begin{pmatrix} 1 & -g \\ e & f^2 - eg \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e & f \end{pmatrix}\begin{pmatrix} 1 & -g \\ 0 & f \end{pmatrix}.$$

Transformation

$$(J^{-1}J_y)_{\bar{y}} + (J^{-1}J_z)_{\bar{z}} = 0.$$

SDYM equation of J-matrix formulation

This equation is so simple! (We will investiagte SDYM in this form)

[10] Y. Brihaye, D.B. Fairlie, J. Nuyts, R.G. Yates, Properties of the self dual equations for an SU(n) gauge theory, J. Math. Phys., 19 (1978) 2528-2532.
[11] K. Pohlmeyer, On the Lagrangian theory of anti-self-dual fields in four-dimensional Euclidean space, Commun. Math. Phys., 72 (1980) 37-47.

Self-dual Yang-Mills equation

- The integrability of J-matrix formulation[12,13]:

$$(\partial_y - \lambda\partial_{\bar{z}})\phi = -J^{-1}J_y\phi, \quad (\partial_z + \lambda\partial_{\bar{y}})\phi = -J^{-1}J_z\phi.$$

- The Miura transformation:

$$J^{-1}J_y = K_{\bar{z}}, \quad J^{-1}J_z = -K_{\bar{y}}.$$

- The K-matrix formulation[14]:

$$K_{y\bar{y}} + K_{z\bar{z}} - [K_{\bar{y}}, K_{\bar{z}}] = 0.$$

→ Generally, we can also call it the SDYM equation

[12] A.A. Belavin, V.E. Zakharov, Yang-Mills equations as inverse scattering problem, Phys. Lett. B, 73 (1978) 53-57.
[13] S.V. Manakov, V.E. Zakharov, Three-dimensional model of relativistic-invariant field theory, inte_x0002_grable by the inverse scattering transform, Lett. Math. Phys., 5 (1981) 247–253.
[14] A.N. Leznov, M.A. Mukhtarov, Deformation of algebras and solution of selfduality equation, J. Math. Phys., 28 (1987) 2574-2578.

Integrable reduction of SDYM

- The SDYM equation is important in integrable system, it possess lots of reduction towards some classical integrable systems.

- Abiowitz et al. had constructed and collected several reductions towards some classical soliton equations[15, 16]:
  - (1 + 1)-dimensions: the KdV equation, the NLS equation, the sine-Gordon equation.
  - (2+1)-dimensions: the KP equation, the Davey–Stewartson equation, the chiral field equations
  - Painléve equations and so on......

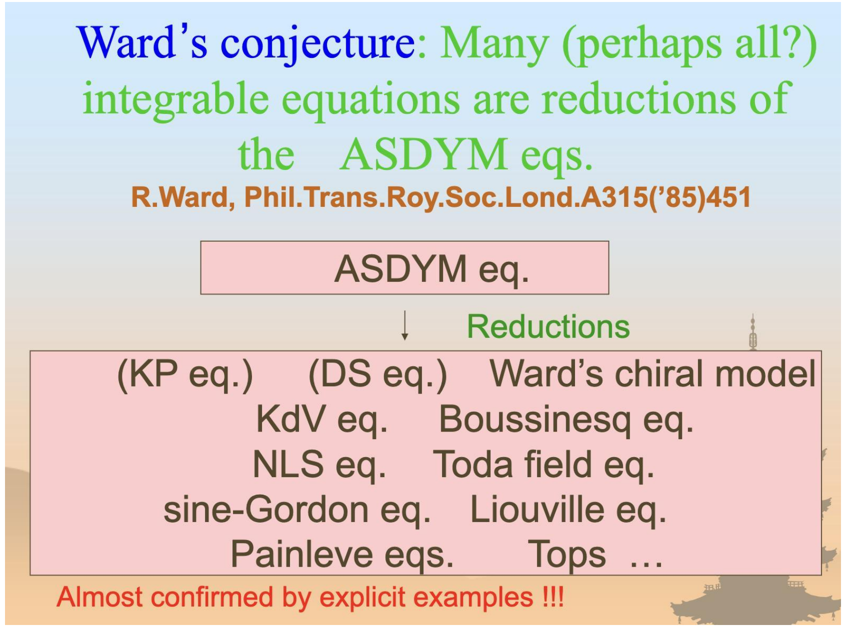
[15] M.J. Abiowitz, S. Chakravarty, L.A. Takhtajan, A self-dual Yang-Mills hierarchy and its reductions to integrable systems in 1+1 and 2+1 dimensions, Commun. Math. Phys., 158 (1993) 289-314.
[16] M.J. Abiowitz, S. Chakravarty, R.G. Halburd, Integrable systems and reductions of the self-dual Yang–Mills equations, J. Math. Phys., 44 (2003) 3147-3173.

Ward conjecture

- There is a famous conjecture proposed by R.S. Ward:

… many (and perhaps all?) of the ordinary and partial differential equations that are regarded as being integrable or solvable may be obtained from the self-dual gauge field equations (or its generalizations) by reduction.

Hamanaka (Nagoya U.) sensei's slides:



Ward's conjecture had been almost confirmed by explicit examples!

SDYM is very important in integrable system theory!

[17] R.S. Ward, Integrable and solvable systems, and relations among them, Philos. Trans. R. Soc. Lond. A, 315 (1985) 451-457.

Cauchy matrix structure of self-dual Yang-Mills equation

Cauchy matrix structure of self-dual Yang-Mills equation

- In our recent research, we worked out the Cauchy matrix structure of self-dual Yang-Mills equation[18,19].

- Starting from certain Sylvester equation and dispersion relations, we derive N-soliton solution formula for unreduced SDYM equation.

- Then we applied suitable reductions to make SDYM equation have physical meaning.

---

[18] S.S. Li, C.Z. Qu, X.X. Yi, D.J. Zhang, Cauchy matrix approach to the SU(2) self-dual Yang-Mills equation, Stud. Appl. Math., 148 (2022) 1703-1721.
[19] S.S. Li, C.Z. Qu, D.J. Zhang, Solutions to the SU(N) self-dual Yang-Mills equation, Physica D, 453 (2023) 133828 (17pp).

Cauchy matrix structure of self-dual Yang-Mills equation

- Assumption1: Sylvester equation:

$$\boldsymbol{K}\boldsymbol{M} - \boldsymbol{M}\boldsymbol{L} = \boldsymbol{r}\boldsymbol{s}^T,$$

→ In this case, r and s are no longer N-th column vectors, but Nx2 order matrices!

- Assumption2: Dispersion relations:

$$\boldsymbol{r}_{x_n} = \boldsymbol{K}^n \boldsymbol{r}\boldsymbol{a}, \quad \boldsymbol{s}_{x_n} = -(\boldsymbol{L}^T)^n \boldsymbol{s}\boldsymbol{a},$$

$a=\mathrm{diag}(a_1,a_2)$ is diagonal matrix

- Assumption3: Master function:

$$\boldsymbol{S}^{(i,j)} = \boldsymbol{s}^T \boldsymbol{L}^j \boldsymbol{M}^{-1} \boldsymbol{K}^i \boldsymbol{r}, \quad i,j \in \mathbb{Z},$$

So that $S^{(i,j)}$ can be a 2X2 matrix

- Lamma1: If Assumption1,2,3 hold, the derivatives of $S^{(i,j)}$ can be calculated as:

$$\boldsymbol{S}^{(i,j)}_{x_n} = \boldsymbol{S}^{(i+n,j)}\boldsymbol{a} - \boldsymbol{a}\boldsymbol{S}^{(i,j+n)} - \sum_{l=0}^{n-1} \boldsymbol{S}^{(n-1-l,j)}\boldsymbol{a}\boldsymbol{S}^{(i,l)}, \qquad n \in \mathbb{Z}^+,$$

$$\boldsymbol{S}^{(i,j)}_{x_0} = \boldsymbol{S}^{(i,j)}\boldsymbol{a} - \boldsymbol{a}\boldsymbol{S}^{(i,j)} = [\boldsymbol{S}^{(i,j)}, \boldsymbol{a}],$$

$$\boldsymbol{S}^{(i,j)}_{x_n} = \boldsymbol{S}^{(i+n,j)}\boldsymbol{a} - \boldsymbol{a}\boldsymbol{S}^{(i,j+n)} + \sum_{l=-1}^{n} \boldsymbol{S}^{(n-1-l,j)}\boldsymbol{a}\boldsymbol{S}^{(i,l)}, \qquad n \in \mathbb{Z}^-.$$

Cauchy matrix structure of self-dual Yang-Mills equation

- Lemma2: If Assumption1,2,3 hold, there is adifference relations for $S^{(i,j)}$:

$$\boldsymbol{S}^{(i+1,j)} - \boldsymbol{S}^{(i,j+1)} = \boldsymbol{S}^{(0,j)}\boldsymbol{S}^{(i,0)}.$$

- Theorem1: If $S^{(i,j)}$ satisfies Lemma1 and Lemma2, by following definition

Then the definition:

$$\boldsymbol{V} \doteq \boldsymbol{I}_2 - \boldsymbol{s}^T\boldsymbol{M}^{-1}\boldsymbol{K}^{-1}\boldsymbol{r}, \qquad \boldsymbol{U} \doteq \boldsymbol{s}^T\boldsymbol{M}^{-1}\boldsymbol{r},$$

yields the following differential recurrence relations:

$$\boldsymbol{V}_{x_{n+1}}\boldsymbol{V}^{-1} = -\boldsymbol{U}_{x_n}, \qquad \boldsymbol{V}_{x_{m+1}}\boldsymbol{V}^{-1} = -\boldsymbol{U}_{x_m}, \qquad n, m \in \mathbb{Z}.$$

The Miura transformation of SDYM equation

Thus $\boldsymbol{V}$ *and* $\boldsymbol{U}$ *solve the* J*- and* K*-matrix formulation SDYM equations respectively:*

$$(\boldsymbol{V}_{x_{n+1}}\boldsymbol{V}^{-1})_{x_m} - (\boldsymbol{V}_{x_{m+1}}\boldsymbol{V}^{-1})_{x_n} = 0, \qquad \boldsymbol{U}_{x_n,x_{m+1}} - \boldsymbol{U}_{x_m,x_{n+1}} - [\boldsymbol{U}_{x_n}, \boldsymbol{U}_{x_m}] = 0.$$

Cauchy matrix structure of self-dual Yang-Mills equation

- Theorem1 is the main result of our paper, it is simple and easy to prove. (Through a direct calculation can verify this result!)

J-matrix formulation: $(\boldsymbol{V}_{x_{n+1}}\boldsymbol{V}^{-1})_{x_m} - (\boldsymbol{V}_{x_{m+1}}\boldsymbol{V}^{-1})_{x_n} = 0,$

Miura transformation: $\boldsymbol{V}_{x_{n+1}}\boldsymbol{V}^{-1} = -\boldsymbol{U}_{x_n}, \quad \boldsymbol{V}_{x_{m+1}}\boldsymbol{V}^{-1} = -\boldsymbol{U}_{x_m},$

K-matrix formulation: $\boldsymbol{U}_{x_n,x_{m+1}} - \boldsymbol{U}_{x_m,x_{n+1}} - [\boldsymbol{U}_{x_n}, \boldsymbol{U}_{x_m}] = 0.$

Our Cauchy matrix structure indicates this! Thus the SDYM (J-/K-formulation) equation can be obtained!

Integrable reduction of SDYM equation

- However, this simple relation provides us a different viewpoint to connect SDYM equation and other integrable systems.

- By taking n=0, the x_0-derivative disappears, and can be represented as a Lie bracket:

$$\boldsymbol{S}^{(i,j)}_{x_0} = \boldsymbol{S}^{(i,j)}\boldsymbol{a} - \boldsymbol{a}\boldsymbol{S}^{(i,j)} = [\boldsymbol{S}^{(i,j)}, \boldsymbol{a}], \qquad \boldsymbol{V}_{x_0} = [\boldsymbol{V}, \boldsymbol{a}], \quad \boldsymbol{U}_{x_0} = [\boldsymbol{U}, \boldsymbol{a}],$$

- This phenomenon indicates us a possible way to reduce the 4d SDYM equation to 2d or 3d classical integrable systems.

Integrable reduction of SDYM equation

- The (-1)st-AKNS equation:

The (-1)st-AKNS equation is an unreduced form of the sine-Gordon equation. It can be derived by setting $n=-1, m=0$ and letting $x=x_1, t=-x_{-1}$ in (8.11), which yields

The reduced K-matrix formulation:

$$\boldsymbol{u}_{xt}+[\boldsymbol{a}+\boldsymbol{u}_t,[\boldsymbol{u},\boldsymbol{a}]]=0, \tag{7.2}$$

$$\boldsymbol{u} \doteq \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

where $\boldsymbol{S}_{x_0}^{(i,j)}=[\boldsymbol{S}^{(i,j)},\boldsymbol{a}]$ has been utilized. Taking $\boldsymbol{a}=\mathrm{diag}(1,-1)$, one can expand equation (7.2) to be:

$$u_{11,xt}-2(u_{12}u_{21})_t=0, \tag{7.3a}$$

$$u_{12,xt}+4u_{12}+4u_{12}(u_{11}-u_{22})_t=0, \tag{7.3b}$$

$$u_{21,xt}+4u_{21}+4u_{21}(u_{11}-u_{22})_t=0, \tag{7.3c}$$

$$u_{22,xt}+2(u_{12}u_{21})_t=0, \tag{7.3d}$$

$u_{11} = -u_{22}$

which reveals a closed system of u_{12} and u_{21} (see (4.11) in [68]):

$$u_{12,xt}+4u_{12}+8u_{12}\int(u_{12}u_{21})_t\mathrm{d}x=0, \tag{7.4a}$$

$$u_{21,xt}+4u_{21}+8u_{21}\int(u_{12}u_{21})_t\mathrm{d}x=0. \tag{7.4b}$$

Integrable reduction of SDYM equation

- The 2nd-AKNS equation:

The 2nd-AKNS equation is an unreduced form of the nonlinear Schrödinger equation. It can be derived by setting $n=1, m=0$ and letting $x=x_1, t=-\mathrm{i}x_2$ in (8.11):

The reduced K-matrix formulation:
$$\boldsymbol{u}_{xx}+\mathrm{i}[\boldsymbol{u}_t,\boldsymbol{a}]-[[\boldsymbol{u},\boldsymbol{a}],\boldsymbol{u}_x]=0. \tag{7.5}$$

Applying $\boldsymbol{S}^{(1,0)}=\boldsymbol{S}^{(0,1)}+\boldsymbol{u}^2$, the derivative $\boldsymbol{u}_x$ can be expressed as

$$\boldsymbol{u}_x=\boldsymbol{S}^{(1,0)}\boldsymbol{a}-\boldsymbol{a}\boldsymbol{S}^{(0,1)}-\boldsymbol{u}\boldsymbol{a}\boldsymbol{u}=(\boldsymbol{S}^{(0,1)}+\boldsymbol{u}^2)\boldsymbol{a}-\boldsymbol{a}\boldsymbol{S}^{(0,1)}-\boldsymbol{u}\boldsymbol{a}\boldsymbol{u}=[\boldsymbol{S}^{(0,1)},\boldsymbol{a}]+\boldsymbol{u}[\boldsymbol{u},\boldsymbol{a}].$$

Thus equation (7.5) can be rewritten as

$$\boldsymbol{u}_{xx}+\mathrm{i}[\boldsymbol{u}_t,\boldsymbol{a}]+[\boldsymbol{u},[\boldsymbol{u},\boldsymbol{a}]][\boldsymbol{u},\boldsymbol{a}]=[[\boldsymbol{u},\boldsymbol{a}],[\boldsymbol{S}^{(0,1)},\boldsymbol{a}]]. \tag{7.6}$$

$$\boldsymbol{u}\doteq\begin{pmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{pmatrix}$$

Taking $\boldsymbol{a}=\mathrm{diag}(1,-1)$, it expands to

$$\begin{pmatrix}u_{11,xx}-4u_{12}u_{21}(u_{11}-u_{22}) & u_{12,xx}-2\mathrm{i}u_{12,t}-8u_{12}^2u_{21}\\ u_{21,xx}+2\mathrm{i}u_{21,t}-8u_{12}u_{21}^2 & u_{22,xx}+4u_{12}u_{21}(u_{11}-u_{22})\end{pmatrix}=\begin{pmatrix}*1 & 0\\ 0 & *2\end{pmatrix}, \tag{7.7}$$

Integrable reduction of SDYM equation

where $*1$ and $*2$ are certain nonzero components. Then components u_{12} and u_{21} form a closed system, referred to as the 2nd-AKNS system (see (3.17) in [68]):

$$u_{12,xx} - 2\mathrm{i}u_{12,t} - 8u_{12}^2u_{21} = 0, \tag{7.8a}$$

$$u_{21,xx} + 2\mathrm{i}u_{21,t} - 8u_{12}u_{21}^2 = 0. \tag{7.8b}$$

The KP equation:

$$\boldsymbol{U}_{x_n,x_{m+1}} - \boldsymbol{U}_{x_m,x_{n+1}} - [\boldsymbol{U}_{x_n}, \boldsymbol{U}_{x_m}] = 0.$$

Let n=2, m=1, then it corresponds to pKP equation

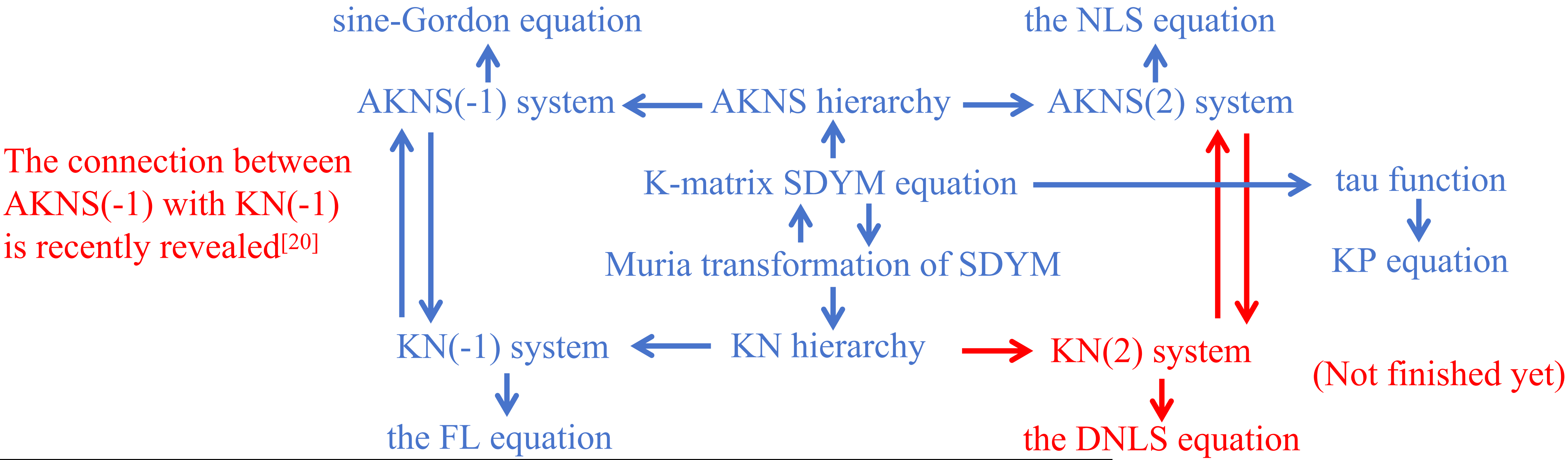
$$\boldsymbol{U} \doteq \boldsymbol{s}^T \boldsymbol{M}^{-1} \boldsymbol{r},$$

$$2u_{11} = Tr(Ua) = Tr(s^T M^{-1} ra) = Tr(M^{-1} M_x) = \frac{|M|_x}{|M|} = \ln(\tau)_x$$

The bilinear transformation for pKP equation

Integrable reduction of SDYM equation

- Recently, we have found a SDYM reduction to the Fokas-Lenells equation, which corresponds to Kaup-Newell spectral problem. (on progress, coming soon!)



[20] R.S. Ye, Y. Zhang, A vectorial Darboux transformation for the Fokas–Lenells system, Chaos Solitons Fractals, 169 (2023) 113233 (7pp).

Thank you for your attentions!

ありがとうございました